# DOCUMENT RESUME

ED 173 130 SE 027 963

| | |
|---|---|
| AUTHOR | Anderson, R. D.; And Others |
| TITLE | Mathematics for Junior High School. Commentary for Teachers. Volume I (Part 1). |
| INSTITUTION | Stanford Univ., Calif. School Mathematics Study Group. |
| SPONS AGENCY | National Science Foundation, Washington, D.C. |
| PUB DATE | 59 |
| NOTE | 284p.; For related documents, see SE 027 964-967 and ED 130 872; Contains occasional light and broken type |
| EDRS PRICE | MF01/PC12 Plus Postage. |
| DESCRIPTORS | Curriculum; *Curriculum Guides; Fractions; *Geometry; *Instruction; Junior High Schools; Mathematics Education; *Measurement; *Number Concepts; Prime Numbers; Secondary Education; *Secondary School Mathematics |
| IDENTIFIERS | *School Mathematics Study Group |

ABSTRACT

This is part one of a two-part manual for teachers using SMSG junior high school text materials. For each chapter, a brief overview is given and sample test questions are listed. Each section contains a discussion of the topic at hand and answers to all the exercises. Chapter topics include: (1) what is mathematics; (2) numeration; (3) whole numbers; (4) non-metric geometry; (5) factoring and primes; (6) the rational number system; and (7) measurement. (MP)

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# MATHEMATICS FOR JUNIOR HIGH SCHOOL

## *Commentary for Teachers*

## VOLUME I (Part 1)

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# MATHEMATICS FOR JUNIOR HIGH SCHOOL

## *Commentary for Teachers*

## Volume I (Part 1)

Prepared under the supervision of the Panel on 7th and 8th Grades of the School Mathematics Study Group:

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*Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.*

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**PRINTED IN THE U.S.A.**

This volume was prepared at a writing session held at the University of Michigan, June 15–August 7, 1959. It is based, in part, on material prepared at the first SMSG writing session, held at Yale University in the summer of 1958. The members of the Ann Arbor writing team were:

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# FOREWORD

The increasing contribution of mathematics to the culture of the modern world, as well as its importance as a vital part of scientific and humanistic education, has made it essential that the mathematics in our schools be both well selected and well taught.

With this in mind, the various mathematical organizations in the United States cooperated in the formation of the School Mathematics Study Group. This Study Group includes college and university mathematicians, high school teachers of mathematics, experts in education, and representatives of science and technology. The general objective of the Study Group is the improvement of the teaching of mathematics in the schools of this country. The National Science Foundation has provided substantial funds for the support of this endeavor.

One of the prerequisites for the improvement of the teaching of mathematics in our schools is an improved curriculum--one which takes account of the increasing use of mathematics in science and technology and in other areas of knowledge and at the same time one which reflects recent advances in mathematics itself. One of the first projects undertaken by the School Mathematics Study Group was to enlist a group of outstanding mathematicians and mathematics teachers to prepare a series of high school textbooks which would illustrate such an improved curriculum. This textbook is based upon 14 experimental units which comprised a first product of this project.

The professional mathematicians in the Study Group believe that the mathematics presented in this text is important for all well-educated citizens in our society to know and that it is also important for the pre-college student to learn in preparation for advanced work in the field. At the same time, the high school teachers in the Study Group believe that it is presented in such a form that it can be readily grasped by students.

In most instances the material presented will have a familiar note to it, but the flavor of presentation, the point of view, as it were, will be different. Some material will be entirely new to the traditional curriculum. This is as it should be, for mathematics is a living and an ever-growing subject, and not a dead and frozen product of antiquity. This healthy fusion of the old and the new should lead a student to a better understanding of the basic concepts and structure of mathematics and provide a firmer foundation for understanding and use of mathematics in a scientific society.

It is not intended that this book be regarded as the only definitive way of presenting good mathematics to students at this level. Instead, it should be thought of as a sample of the kind of improved curriculum that we need and as a source of suggestions for the authors of the commercial textbooks of the future. It is sincerely hoped that these texts will lead the way toward inspiring a more meaningful teaching of Mathematics, the Queen and Servant of the Sciences.

# TABLE OF CONTENTS

Page

Chapter 1 What Is Mathematics?................ 1

Chapter 2 Numeration.......................... 15

Chapter 3 Whole Numbers...................... 63

Chapter 4 Non-Metric Geometry................. 85

Chapter 5 Factoring and Primes................ 125

Chapter 6 The Rational Number System I........ 163

Chapter 7 Measurement......................... 201

## NOTE TO TEACHERS

Although some of the material in Volume I has been tried out by junior high school teachers, there is much material that is new. Thus it is difficult to tell how much time will be needed for the different chapters. Estimates by the authors (including junior high school teachers) of a reasonable amount of time for each chapter are as follows:

| Chapter | Approximate Number of Days |
|---|---|
| 1 | 7 |
| 2 | 11 |
| 3 | 14 |
| 4 | 15 |
| 5 | 10 |
| 6 | 20 |
| 7 | 27 |
| 8 | 23 |
| 9 | 10 |
| 10 | 13 |
| 11 | 10 |
| 12 | 10 |
| 13 | 4 |
| Total: | 174 Days |

Teachers are urged to try not to exceed these approximate time allotments so that pupils will not miss the chapters at the end of the course. Some classes will be able to finish certain chapters in less than the estimated time. In any event it is important to be roughly halfway through Chapter 7 at the end of the first semester.

Problems, topics, and sections which were designed for the better students are indicated by an asterisk (*). The starred sections should be used or omitted as a means of adjusting the approximate time schedule.

# PREFACE

Fourteen experimental units for use in the seventh and eighth grades were written in the summer of 1958 and tried out by approximately 100 teachers in 12 centers in various parts of the country in the school year 1958-59. Some of the chapters of this book are quite similar to the corresponding experimental units, and in others there are some important changes. Several of the chapters are entirely new. The changes and additions are based both on the teachers' comments, on their experience in teaching the experimental units, and on knowledge of modern needs. The materials are also carefully chosen to provide adequate preparation for the text materials prepared for use in grades 9 through 12 by SMSG.

Big ideas of junior high school mathematics, emphasized in this text are: structure of arithmetic from an algebraic viewpoint; the real number system as a progressing development; metric and non-metric relations in geometry. Throughout the materials these ideas are associated with their applications. Important at this level are experience with and appreciation of abstract concepts, the role of definition, development of precise vocabulary and thought, experimentation, and proof. Substantial progress can be made on these concepts in the junior high school.

Mathematics is fascinating to many persons because of its opportunities for creation and discovery as well as for its utility. It is continuously and rapidly growing under the prodding of both intellectual curiosity and practical applications. Even junior high school students may formulate mathematical questions and conjectures which they can test and perhaps settle; they can develop systematic attacks on mathematical problems whether or not the problems have routine or immediately determinable solutions. Recognition of these important factors has played a considerable part in selection of content and method in this text.

We firmly believe mathematics can and should be studied with success and enjoyment. It is our hope that this text may greatly assist all teachers who use it to achieve this highly desirable goal.

## CHAPTER 1

# WHAT IS MATHEMATICS?

### General Remarks

This chapter is intended to give the pupil an appreciation for the importance of mathematics. Its objectives are:

I. To develop an understanding of what mathematics is as opposed to simple computation.

II. To develop an appreciation of the role of mathematics in our culture.

III. To motivate pupils by pointing out the need for mathematicians and for mathematically trained people.

Since this chapter is much different from ordinary textbook material it will need a different treatment. The purpose of the chapter is not to teach many facts or skills, but rather to build an enthusiasm for the study of mathematics. Good attitudes will be built if you use imagination and enthusiasm in getting these objectives across to the pupils.

It is expected that from six to eight lessons will be sufficient for this chapter. Certainly no more than eight days should be devoted to it.

One way of presenting the latter part of the chapter (Sections 1-4 through 1-9) is to let the students read the sections at home. Follow this with daily discussions on each section and some work on the exercises.

Note that Exercises 1-6 and Exercises 1-7 are suggestions for background study to be carried on throughout the year. These should be begun during the first week, with periodic reports on progress by pupils.

It might be worthwhile to have the pupils read this chapter again at the end of the year. The problems might also be solved again. They should be much easier to solve after the course has been completed.

Encourage the more able students to solve the brainbusters but be ready to help them if they have difficulties. Most pupils will want to puzzle over the brainbusters for a few days. For this reason, only individual help is suggested until the time seems appropriate for general class discussion.

## 1-1. Mathematics as a Method of Reasoning

It might provide additional challenge to emphasize to the pupils that Exercises 1-1 and Exercises 1-2 are not easy. Moreover, no simple formula for solution can be given. Some of the pupils (and many parents!) will certainly find the problems difficult and time-consuming at this stage. You may not wish to assign all the problems in these two sections.

### Answers to Exercises 1-1

1. Two sons cross; one returns. Father crosses; other son returns. Two sons cross.

2. No. They need a boat carrying 225 pounds. Solution as in 1 above.

3. Man takes goose and returns alone. He takes fox and returns with goose. He takes corn across river and returns alone to pick up goose.

4. Yes. This depends on the fact that $8x + 5y = 2$ has solutions in integers, such as $x = -1$, $y = 2$ and $x = 4$, $y = 6$. The first means that if you fill the 5-gallon jug twice and empty it once into the 8-gallon jug, you will have 2 gallons left. The second solution means that if you fill the 8-gallon jug four times and use it to fill the 5-gallon jug 6 times, you will have 2 gallons left. Point out that the first solution is best.

5. Most pupils will need paper and pencil for this one. As they attempt the solution in front of the class it is a good idea to minimize the help from others in the class. If an error is made, another student should be selected to present his solution. It should be pointed out that a crossing necessitates a landing, i.e. jumping from boat onto boat is not considered ethical.

$M_1$ $M_2$ $M_3$ $C_1$ $C_2$ $C_3$

$M_1$ $C_1$ cross river; $M_1$ returns.
$C_2$ $C_3$ cross river; $C_1$ returns.
$M_1$ $M_2$ cross river; $M_1$ $C_2$ return.
$M_1$ $M_3$ cross river; $C_1$ returns.

$C_1$ $C_2$ cross river; $C_1$ returns

$C_1$ $C_3$ cross river.

6. Balance the two groups of 3 marbles each. If they balance then it is only necessary to balance the remaining two marbles to find the heavy one. If the two groups of 3 marbles do not balance, take the heavier group. Of the 3 marbles in the heavier group balance any 2 marbles. If they balance the remaining marble is the heaviest one. If the 2 marbles do not balance the heaviest will be 1 of the 2 on the balance.

7. In solving the problem is it practical to try out all the possible ways the dominoes may be placed on the board? This would be difficult because there are more than 65,536 ways to cover the whole board. The solution may be found in another way:

   How many squares are there altogether on the board? (64)

   How many squares must be covered? (62)

   What is special about the two squares next to each other? (They are of different colors.)

   What is special about the two opposite corners? (They are the same color.)

   If you place any number of dominoes on the board can you say anything about the kinds of squares which will be covered? How does this compare with the kinds of squares which you are supposed to cover? Do you have to make even one experiment in order to get the answer to the original problem? Can two squares of the same color be covered with one domino? The answer to this question should help some of the students reason why the solution is impossible.

## 1-2. Deductive Reasoning

After the concept of deductive reasoning has been introduced, it is still necessary to give the students an understanding of its importance. Examples of scientific advances in which deductive reasoning was used might be Einstein's discovery of the atomic energy formula $E = MC^2$, space travel, satellites, digital computer development, nuclear energy, etc.

### Answers to Exercises 1-2

1. O.K.
2. No. Yes. Someone gets at least 3 pencils.
3. 367
4. 1. Tell the class that the first two to enroll might be twins.
5. 6
7. Only 1
8. 6 committees
9. 12 committees

## 1-3. From Arithmetic to Mathematics

John Friedrich Karl Gauss was born in Brunswick, Germany, in 1777. He died in 1855 at the age of 78. The pupils may be interested in noting that his lifetime almost spanned the years from the American Revolution to the Civil War.

Many mathematicians consider Gauss as one of the three greatest mathematicians of all times, the other two being Newton and Archimedes.

In this age of space exploration it is interesting to note that Gauss developed powerful methods of calculating orbits of comets and planets. His interests extended also to such fields as magnetism, gravitation, and mapping. In 1833 Gauss invented the electric telegraph, which he and his fellow worker, Wilhelm Weber, used as a matter of course in sending messages.

In 1807 Gauss was appointed Director of the Göttingen Observatory and Lecturer of Mathematics at Göttingen University. In later years the greatest honor that a German mathematician could have was to be appointed to the professorship which Gauss had once held.

This section deals with Gauss's discovery of the known method of summing an arithmetic series. It dramatizes how some pupils (and mathematicians) apply insight to finding a solution to a problem. Your better students should be told that there are methods other than Gauss's for finding the sum of a series of numbers. Some students might be encouraged to discover methods of their own for adding number series quickly.

The "middle number" method is one that may be used. This scheme can be used for an even or an odd number of integers. The following examples may be used to explain this method to the students who have tried to discover other methods.

Example A. $1 + 2 + 3 + 4 + 5 + 6 + 7 = ?$

In this series the middle number (4) is the average of the individual numbers of the series. The sum is the product of the middle number (4) and the number of integers in the series or $4 \times 7 = 28$.

Some pupils may prefer to think of the series as

$$(1 + 7) + (2 + 6) + (3 + 5) + 4 = (4 + 4) + (4 + 4) + (4 + 4) + 4 = 7 \times 4 = 28.$$

Example B. $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = ?$

In this case the "middle number" is half way between 4 and 5, or $4\frac{1}{2}$. Then the product $(4\frac{1}{2}) \times 8 = 36$ is seen to give the correct sum.

It may seem more plausible here to write the sum as

$$(1 + 8) + (2 + 7) + (3 + 6) + (4 + 5) =$$

$$(4\frac{1}{2} + 4\frac{1}{2}) + (4\frac{1}{2} + 4\frac{1}{2}) + (4\frac{1}{2} + 4\frac{1}{2}) + (4\frac{1}{2} + 4\frac{1}{2}) = 8 \times 4\frac{1}{2}$$

Clearly, Gauss's method is to be preferred in this case.

## Answers to Exercises 1-3

1. Another method is this: $2 + 4 = 3 + 3$, $1 + 5 = 3 + 3$.

   That is, the sum is the same as:

$3 + 3 + 3 + 3 + 3 = 5 \times 3 = 15$

This can be called the "averaging method."

2. Either method works. Gauss method: $\frac{8 \times 5}{2}$;

   Averaging method: $5 \times 4 = 20$.

3. $\frac{16 \times 8}{2} = 64$ Here there is an even number of quantities so that the "averaging method" must be modified to give 8 eight's or $8 \times 8 = 64$.

4. $\frac{24 \times 6}{2} = 72$; $6 \times 12 = 72$

5. $\frac{62 \times 10}{2} = 310$; $31 \times 10 = 310$

6. $\frac{50 \times 51}{2} = 1275$; $51 \times 25 = 1275$

7. Yes, provided that the numbers are in arithmetic progression; that is, there is the same difference between each pair of numbers.

8. Yes. If we start with 1 there are 200 integers in the series giving us $\frac{(1 + 200)200}{2}$. If we start with 0 there are 201 integers in the series giving us $\frac{(0 + 200)201}{2}$ The products of like factors are equal. The method also may be used in a series if we select a number other than 1 or 0 as starting points. Some of the better students may investigate whether the method works in other number series.

9. (a) If you add 1 to the quantity, the sum up to any number is equal to the next number. Hence, the sum plus 1 is equal to $2 \cdot 256 = 512$. Therefore, the sum is 511.

(b) This is more in the spirit of Gauss:

$\text{Sum} = 1 + 2 + 4 + \ldots + 256$

$2 \times \text{sum} = 2 + 4 + \ldots + 256 + 512$

Subtracting: $\text{Sum} = 511$

10. $\text{Sum} = 2 + (6 + 18 + \ldots + 486)$

$3 \times \text{sum} = 6 + 18 + \ldots + 486 + 1458$

Subtracting: $2 \times (\text{the sum}) = 1458 - 2 = 1456$

Sum, 728.

## 1-4. Kinds of Mathematics

Discussion of this section should emphasize the dynamic character of mathematics. It is not a "dead" subject as many parents believe.

It is important also to point out here (and throughout the course) that certain important ingredients are common to all the many varieties of mathematics. The method of logical reasoning, the use and manipulation of abstract symbols, the insistence on precision of thought and clarity of expression, the emphasis on general results--these are some characteristics which need to be stressed whenever possible.

### Probability

This section gives only a very brief introduction to probability. Although students may become interested at this point and attempt more complex problems, it would be better if they waited. A supplementary unit on probability comes later in the text.

## Answers to Exercises 1-4

2. One out of four or $\frac{1}{4}$   3. One out of two or $\frac{1}{2}$

4. One out of 52 or $\frac{1}{52}$   5. Four out of 52 or $\frac{4}{52} = \frac{1}{13}$

The probability may be thought of as a ratio of

$$\frac{\text{the number of possible favorable selections}}{\text{the total number of all possible selections.}}$$

The pupils should be reminded that to say his chances are 1 out of 13 does not mean that he will necessarily draw an ace in the first 13 draws.

6. One out of six. A die has 6 sides and only one side has two dots.

7. There are four possibilities in all, only one of which is favorable. Hence the probability is $\frac{1}{4}$.

8. 1 out of 36. The more advanced students should reason that there are 36 possible combinations. A table may be constructed to show the possibilities. A possible diagram is the following.

| Number of Possibilities | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 ... | 34 | 35 | 36 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1st die | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 ... | 6 | 6 | 6 |
| 2nd die | 1 | 2 | 3 | 4 | 5 | 6 | 1 | 2 | 3 ... | 4 | 5 | 6 |

Since there is only one possible way of making two ones the probability is $\frac{1}{36}$.

9. The possibilities are easily enumerated.

| Number of Possibilities | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1st coin | H | H | H | H | T | T | T | T |
| 2nd coin | H | H | T | T | H | H | T | T |
| 3rd coin | H | T | H | T | H | T | H | T |

For 3 heads to come up the probability is 1/8. For exactly 2 heads the probability is 3 out of 8 or 3/8. For at least two heads the chance is 4 out of 8 or 1/2. Note that this equals the probability of exactly two heads (3/8) plus the probability of 3 heads (1/8).

## Exercises 1-6 and Exercises 1-7

The exercises suggested in sections 1-6 and section 1-7 should be undertaken as year-long projects to be reported on periodically. Much of this information would be good bulletin board material. A general aim is to make the pupils alert to the current news relating to mathematics and mathematicians. As the year progresses they should gain an increasing appreciation of the important role mathematical thought is playing in our present civilization.

A survey of college requirements in the vocations they may choose should be especially valuable at this stage. We hope it may ease the difficult task of effective guidance in choosing their high school courses.

One word of caution is perhaps warranted here. The purpose of these sections on mathematics today is to call attention to its central role in the pupil's daily life. The necessity for a minimum knowledge of mathematics is to be stressed. It is not the intention to recruit people for careers as mathematicians.

## 1-8 Mathematics for Recreation

Bring out the fascination of mathematics as a leisure activity or hobby. Encourage students in finding recreational mathematics from the books available at school and from current magazines or rotogravure sections of newspapers.

Choose such problems now and then, throughout the year, at a time when the class needs a change of pace. These kinds of problems can be used profitably with the class period before a lengthy vacation.

### Answer to Exercise 1-8

1. The first figure can be drawn if you start at either of the vertices where an odd number of segments come together. The second figure has no such vertices so it can be drawn by starting at any vertex. The third figure has four vertices where an odd number of segments come together so it cannot be drawn without lifting your pencil or retracing a line.

## Recreational Books

Adler, Irving. MAGIC HOUSE OF NUMBERS. New York: Signet Key Book, The New American Library, Inc. Paperbound 50¢.

Adler, Irving. MATHEMATICS, THE STORY OF NUMBERS, SYMBOLS, AND SPACE. New York: Golden Press, 1958. Library edition $1.39. Paperbound 50¢.

Merrill, Helen A. MATHEMATICAL EXCURSIONS. New York: Dover Publications, 1957. Paperbound $1.00.

Mott-Smith, Geoffrey. MATHEMATICAL PUZZLES FOR BEGINNERS AND ENTHUSIASTS. New York: Dover Publications, Inc., 1954. Paperbound $1.00.

Smith, David Eugene and Ginsburg, J. NUMBERS AND NUMERALS. Washington, D. C.: National Council of Teachers of Mathematics. 35¢.

Wylie, C. R. 101 PUZZLES IN THOUGHT AND LOGIC. New York: Dover Publications, Inc., 1957. Paperbound $1.00.

## 1-9. Highlights of First-Year Junior High Mathematics

This section is intended to give the student a general overview of the scope of the first-year course in junior high mathematics. It is intended to broaden his ideas of mathematics by discussing very briefly the topics that will be studied. This, together with the introduction of logical reasoning and probability earlier in this chapter, will help dispel the general idea that mathematics is only computation.

It may strengthen the student's understanding of what mathematics is to read this chapter again later in the year.

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1. Bell, Eric T. MEN OF MATHEMATICS. New York: Simon and Schuster, 1937.

2. Boehm, George A. W. and the Editors of Fortune. THE NEW WORLD OF MATH. New York: The Dial Press, 1959.

   See especially pp. 17-46 (The New Mathematics).

3. Courant, Richard and Robbins, Herbert. WHAT IS MATHEMATICS? New York: Oxford University Press, 1943.

4. Kline, Morris. MATHEMATICS IN WESTERN CULTURE. New York: Oxford University Press, 1953.

   See especially pp. 3-12 (General Concepts), pp. 21-39 (Deductive Reasoning), pp. 359-369 (Probability).

5. Sawyer, W. W. PRELUDE TO MATHEMATICS. Baltimore, Maryland: Penguin Books, 1957.

## CHAPTER 2

## NUMERATION

### Introduction

For this unit little background is needed except familiarity with the number symbols and the basic operations with numbers. The purpose of the unit is to deepen the pupil's understanding of the decimal notation for whole numbers, especially with regard to place value, and thus to help him delve a little deeper into the reasons for the operations, which he already knows, for addition and multiplication. One of the best ways to accomplish this is to consider systems of number notations using bases other than ten. Since, in using a new base, the pupil must necessarily look at the reasons for "carrying" and the other mechanical operations in a new light, he should gain deeper insight into the decimal system. A certain amount of computation in other systems is necessary to fix these ideas, but such computation should not be regarded as an end in itself. Some of the pupils, however, may enjoy developing a certain proficiency in using new bases in computing.

Perhaps the most important reason for introducing ancient symbolisms for numbers is to contrast them with our decimal system, in which not only the symbol, but its position, has significance. It should be shown, as other systems are presented, that position has some significance in them also. The Roman system has a start in this direction in that XL represents a different number from LX, but the start was a very primitive one. The Babylonians also made use of position, but lacked a symbol for zero until about 200 B.C. The symbol "$\overset{<}{<}$" denoted the absence of a figure but apparently was not used in computation. The numeral zero is necessary in a positional system. In order for the pupils to appreciate the important characteristics of our system of writing numbers, the following table may be discussed.

| | Base | Place Value | Zero |
|---|---|---|---|
| Egyptian | Ten | No | No |
| Babylonian | Sixty | Yes | Limited meaning |
| Roman | Varied | No, but it has positional value | No |
| Decimal | Ten | Yes | Yes |

Pupils should not be expected to memorize ancient symbolism. It is recommended that very little time be spent on the use of the symbols themselves.

It is especially important to distinguish between a number and the symbols by which it is represented. Some of the properties usually connected with a number are really properties of its notation. The facts that, in decimal notation, the numeral for a number divisible by 5 ends in 5 or 0, and that $\frac{1}{3}$ has an unending decimal equivalent are illustrations. Most of the properties with which we deal are properties of the numbers themselves, and are entirely independent of the notation in which they are represented. Examples of such properties of numbers are: $2 + 3 = 3 + 2$; the number eleven is a prime number; and six is greater than five. The distinction between a number and the notation in which it is expressed should be emphasized whenever there is opportunity.

An attempt has been made to use "number" and "numeral" with precise meaning in the text. For example, "numerals" are written, but "numbers" are added. A numeral is a written symbol. A number is a concept. Later in the text it may be cumbersome to the point of annoyance to speak of "adding the numbers represented by the numerals written below." In such case the expression may be elided to "adding the numbers below."

At several points, numbers are represented by collections of x's. Exercises of this kind are important, because they show the role of the base in grouping the x's, as well as the significance of the digits in the numeral for the number.

Your attention is called to the end of the commentary for this chapter, where you will find a glossary of terms as they are used in this chapter and a bibliography for further reference.

## Suggested Time Schedule

It is important that only enough time be spent on the various sections to secure the understandings desired. The historical symbols themselves are not important. Neither are the numerals in other bases valuable in themselves, but the ideas that they help to clarify are important. Hence, each section was planned to take one lesson, except for Section 5 which was planned for two lessons. Twelve days should be sufficient time for the unit and a test on the unit.

Familiarity with the subject matter is an important factor in a smooth presentation. Teachers report that a second experience with this material is much easier than the first. The lesson moves more rapidly, apparently, as the teacher gains confidence in the subject matter presented.

Homogeneously grouped classes undoubtedly will alter the suggested schedule since the more able students can complete the chapter in less time while less able students may require a considerably longer period of time on various sections. The following schedule may then be adapted to local needs, taking into consideration the length of class periods, and other factors. It should be remembered that extra time spent on this chapter will necessarily reduce the number of days available for later important chapters.

| Sections | Days |
|---|---|
| 1 | 1 |
| 2 | 1 |
| 3 | 1 |
| 4 | 1 |
| 5 | 2 |
| 6 | 1 |
| 7 | 1 |
| 8 | 1 |
| 9 | 1 |
| 10 | 1 |
| Test | 1 |
| Total | 12 |

## 2-1. History of Numerals

The purpose of the historical material is to trace the continuing need for convenient symbols and for a useful way of writing expressions for numbers. The idea of "one-to-one" correspondence is introduced. It is developed later and should not be defined here where the emphasis is upon numerals rather than upon number. Egyptian symbolism is introduced to familiarize the pupils with one of the first important systems of notation. Do not use an excessive amount of time in discussing the Egyptian or Babylonian systems.

The Babylonians were among the first to use place value. The base sixty system is mentioned because of later references to it, particularly in measurement. There is evidence that the Babylonians also used symbols like○◗but there is no need to introduce these to the pupils. No pupil should be required to memorize ancient symbolism except in the case of Roman numerals.

The Roman system may be stressed because of its continued use. Note that the subtracting principle was a late development. It may be pointed out that computation in ancient symbolism was complex and sometimes very difficult. Because of this, various devices were used, such as the sand reckoner, counting table, and abacus. After decimal numerals became known, algorithms were devised and people were able to calculate with symbols alone. There was much opposition in Europe to the introduction and use of Hindu-Arabic numerals, especially on the part of the abacists. As the new system became accepted, the abacus and other computing devices slowly disappeared in Europe.

## Answers to Exercises 2-1

1.

|  | a | b | c |
|---|---|---|---|
| Egyptian | ∩∩∩ \|\|\|<br>∩∩ | ∩∩∩∩<br>∩∩∩ | 𓍢∩\| |
| Babylonian | <<<<< 𒁹𒁹𒁹 | 𒁹< | 𒁹<<<<< 𒁹 |

2. a. 671 b. 1029 c. 1,100,200

3. ∩∩||, or ||∩∩ or ∩|∩| or |∩|∩, etc.

4. No. < 𒁹 means (10 + 1), or 11. It could also mean (600 + 1), or 601, etc. 𒁹< means (60 + 10), or 70. It could also mean (3600 + 10), or 3610, etc.

5. a. 𒁹𒁹 could mean (1 + 1), or 2, or it could mean (60 + 1), or 61, or it could mean (3600 +1), or 3601, etc.

   b. No. "11" means (1 × 10) + (1 × 1) = 10 + 1.

6. a. LI b. XIX c. CLX d. DCCCLXXXVIII

   e. MCMLIX f. $\overline{\text{M}}$

7. a. 29 b. 61 c. 90 d. 105 e. 666

   f. 500,000

8. A "one-to-one correspondence" can be established between the red cards and the black cards in the deck. It might be explained to the pupils that this method was the same

as that employed by the shepherd who counted his sheep by comparing the number of sheep with a collection of pebbles.

## 2-2. Reading and Writing Numerals in the Decimal System

The illustration of grouping in tens suggests a method useful in mental calculation as 37 + 62 is 3 tens + 6 tens + 7 + 2 which is 99. Note that parentheses are used to show that certain combinations are to be considered as representing a single numeral. In a later chapter a different use of parentheses will be discussed in greater detail. Emphasize the value represented by a digit and the value of position in decimal notation.

Emphasize the importance of the invention of a useful system which lends itself easily to calculation. Its efficiency lies in the small number of symbols used, with no need for new or additional symbols as larger numbers are introduced; in the use of place value where each position corresponds to a power of the base; and in the use of zero as a place holder and in computation.

The reading and writing of numerals may be treated as a review, or if needed, as a thorough study, depending on the needs of pupils. Some pupils may know and understand this material completely. Others may have a very limited proficiency in this area.

A class discussion might deal with the following:

Suppose we used systematic names for numerals such as "two tens" for twenty, "two tens, one" for twenty-one, "ten, one" for eleven, and "ten, two" for twelve. How many different, basic words would be needed to name all counting numbers up to a trillion? Remember that a numeral like "one hundred, three tens, six" is made up of basic words used many times in other numerals.) There are fifteen essential words: "one, two, three,......, ten, hundred, thousand, ... trillion."

## Answers to Exercises 2-2

1. Ten
2. (1) one, (2) ten, (3) hundred, (4) thousand, (5) ten thousand, (6) hundred thousand, (7) million, (8) ten million, (9) hundred million.
3. Following billion are trillion, quadrillion, quintillion, sextillion, septillion, octillion, nonillion, decillion, undecillion, duodecillion, tredecillion, quattuordecillion, quindecillion, sexdecillion, septendecillion, octodecillion, novemdecillion, and vigintillion. (The last represents a numeral "1" followed by 63 zeros.)
4. 99,999 means (9 × 10,000) + (9 × 1000) + (9 × 100) + (9 × 10) + (9 × 1)

   ninety-nine thousand, nine hundred ninety-nine
5. 100,000 means (1 × 100,000) + (0 × 10,000) + (0 × 1000) + (0 ×100) + (0 × 10) + (0 × 1). One hundred thousand.
6. 1,000,000 means (1 × 1,000,000) + (0 × 100,000) +

(0 × 10,000) + (0 × 1000) + (0 × 100) + (0 × 10) + (0 × 1). One million.

7. a. three hundred.
   b. three thousand five.
   c. seven thousand one hundred nine.
   d. fifteen thousand, fifteen.
   e. two hundred thirty-four thousand.
   f. six hundred eight thousand, fourteen.
   g. one hundred thousand, nine
   h. nine hundred ninety-nine thousand, nine hundred ninety-nine.

   (Note: Only the tens numbers are hyphenated, as "twenty-three, etc.")

8. a. seven million, thirty-six thousand, two hundred ninety-eight.
   b. nine trillion, three hundred billion, seven hundred eight million, five hundred thousand.
   c. twenty billion, three hundred million, four hundred thousand, five hundred.
   d. nine hundred billion.

9. a. 159
   b. 503
   c. 6857
   d. 3,070,013

e. 4,376,007,000

f. 20,010

t. 9,015,200

10. It would be easier to write a numeral for the larger number. There are numbers which, even though they have relatively simple names in symbols, do not have convenient word names at all. There has been no real need to have word names beyond "novemdecillion" and "vigintillion."

## 2-3. Expanded Numerals and Exponential Notations

Exponents are introduced here in a situation which shows clearly their usefulness for concise notation. Furthermore, their use serves to emphasize the role of the base and of position. This role will be more fully utilized in the sections to follow.

The books listed in the bibliography give good additional information about various systems of numeration.

The Celts and Mayans used twenty as a base probably because they used their toes as well as their fingers in counting. The special names sometimes used for twenty is "score." Some Eskimo tribes probably count by five using the fingers of one hand.

### Answers to Exercises 2-3

1. Ten to the first power, ten to the second power (or ten square), ten to the third power, (or ten cube), ten to the fourth power, ten to the fifth power, ten to the

sixth power, ten to the seventh power, ten to the eighth power, ten to the ninth power, ten to the tenth power.

2. a. $3^5$
   b. $2^4$
   c. $6^6$
   d. $25^3$
   e. $5^6$
   f. $4^2$
   g. $279^5$
   h. $16^1$

3. a. three
   b. seven
   c. two
   d. ten
   e. As used here the exponent indicates the value of the base, five.
   f. five

4. a. 4 × 4 × 4
   b. 3 × 3 × 3 × 3
   c. 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2
   d. 10 × 10 × 10 × 10 × 10 × 10 × 10
   e. 33 × 33 × 33 × 33 × 33
   f. 175 × 175 × 175× 175 × 175 × 175

5. The exponent tells how many times the base is taken as a factor.

6. a. 27
   b. 25
   c. 256
   d. 32
   e. 36
   f. 343
   g. 64
   h. 81
   i. 1000
   j. 81
   k. 64
   l. 1024

7. a. $4^3$ means 64. $3^4$ means 81.
   b. $2^9$ means 512. $9^2$ means 81.

8. a. $(4 \times 10^2) + (6 \times 10^1) + (8 \times 1)$

   b. $(5 \times 10^3) + (3 \times 10^2) + (2 \times 10^1) + (4 \times 1)$

   c. $(7 \times 10^3) + (0 \times 10^2) + (6 \times 10^1) + (2 \times 1)$

   d. $(5 \times 10^4) + (9 \times 10^3) + (1 \times 10^2) + (2 \times 10^1) + (6 \times 1)$

   e. $(1 \times 10^5) + (0 \times 10^4) + (9 \times 10^3) + (1 \times 10^2) + (8 \times 10^1) + (0 \times 1)$

| | | | |
|---|---|---|---|
| 9. | $10^{10}$ | 10,000,000,000 | ten billion |
| | $10^9$ | 1,000,000,000 | one billion |
| | $10^8$ | 100,000,000 | one hundred million |
| | $10^7$ | 10,000,000 | ten million |
| | $10^6$ | 1,000,000 | one million |
| | $10^5$ | 100,000 | one hundred thousand |
| | $10^4$ | 10,000 | ten thousand |
| | $10^3$ | 1,000 | one thousand |
| | $10^2$ | 100 | one hundred |
| | $10^1$ | 10 | ten |

10. a. $10^3$ b. $10^5$ c. $10^6$ d. $10^8$

11. 100, 10, 1. Some discussion might be devoted to the meaning given to $10^0$. This point need not be stressed at this time, but it will be used in later chapters.

12. The exponent of the base "10" tells how many zeros are written to the right of the "1" when the numeral is written in the usual way.

13. $10^{100}$. (It may be pointed out to the pupils that $1^{100}$ is 1.)

## 2-4. Numerals in Base Seven

The purpose of teaching systems of numeration on bases other than ten is not to produce facility in calculating with such systems. A study of an unfamiliar system aids in understanding a familiar one, just as the study of a foreign language aids us in understanding our own. The decimal system is so familiar that its structure and the ideas involved in its algorithms are easily overlooked. In this section attention is focused on numerals, rather than on numbers.

Questions may arise about the notation for a numeral to base seven. We do not write "$15_7$" because the symbol "7" does not occur in a system of numeration to this base. Replacing the numeral by the written word emphasizes this fact.

Later in the chapter and in succeeding chapters, some classes may agree to indicate the base, as different systems of numeration are introduced, by a numeral in decimal notation. They may agree that they will regard the subscript in this case as always based on ten. Thus they may write for $25_{seven}$ the expression $25_7$; for $18_{twelve}$ the expression $8_{12}$; and for $110_{two}$, $110_2$.

After the pupils have had practice in grouping by sevens, introduce counting. Pupils enjoy counting in turn, and helping each other as $30_{seven}$, $40_{seven}$, and like numerals arise. Have them fill in the missing parts of the list on page 39 orally with perhaps a recorder at the board.

Numerals (in base seven) from $21_{\text{seven}}$ to $66_{\text{seven}}$

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 30 | 40 | 50 | 60 | 100 | 110 | 120 | 130 | 140 | 150 | 160 | 200 |
| 21 | 31 | 41 | 51 | 61 | 101 | 111 | 121 | 131 | 141 | 151 | 161 | 201 |
| 22 | 32 | 42 | 52 | 62 | 102 | 112 | 122 | 132 | 142 | 152 | 162 | 202 |
| 23 | 33 | 43 | 53 | 63 | 103 | 113 | 123 | 133 | 143 | 153 | 163 | |
| 24 | 34 | 44 | 54 | 64 | 104 | 114 | 124 | 134 | 144 | 154 | 164 | |
| 25 | 35 | 45 | 55 | 65 | 105 | 115 | 125 | 135 | 145 | 155 | 165 | |
| 26 | 36 | 46 | 56 | 66 | 106 | 116 | 126 | 136 | 146 | 156 | 166 | |

At $66_{\text{seven}}$ you may wish to say: "This is the number of states we had in the United States before Alaska became a state. How many states did we have after Alaska and before Hawaii? How shall we write this number in base seven numerals? We have gone as high as we can in the 'one' place and in the 'seven' place. What is one more than $66_{\text{seven}}$? What do you do when you reach 99 in the decimal system?" When the pupils understand that after $66_{\text{seven}}$ comes $100_{\text{seven}}$, ask them, "How many states are there when we include Hawaii?" Pupils may read $101_{\text{seven}}$ as "one, oh, one, base seven."

Have the pupils continue to count orally until they reach $202_{\text{seven}}$.

It is usually helpful to keep the chart on page 39 on the board during the time this section is studied. Some teachers emphasize the meaning of exponents by writing the chart in two ways:

| | | | |
|---|---|---|---|
| $\text{seven}^4$ | | $\text{seven}^3$ | |
| (seven × seven × seven × seven) | | (seven × seven × seven) | |
| $\text{seven}^2$ | seven | | one |
| (seven × seven) | (seven) | | (one) |

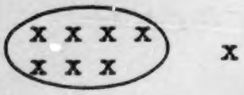
It is suggested that alternate exercises in this list be discussed and answered in class as a group undertaking. The pupils should then be ready to attempt the remaining exercises without further help.

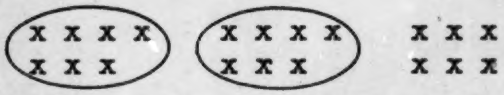
## Answers to Exercises 2-4

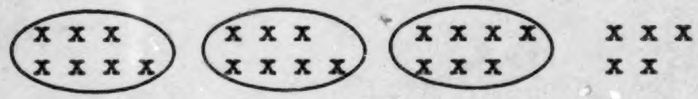
1. a. $13_{seven}$

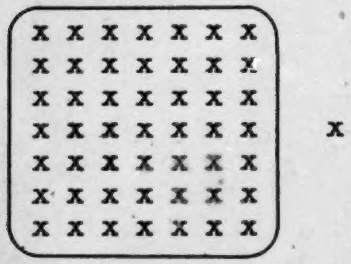
   b. $24_{seven}$

   c. $116_{seven}$

2. a. 

   b. 

   c. 

   d. 

3. a. $(3 \times \text{seven}) + (3 \times \text{one}) = 24$

   b. $(4 \times \text{seven}) + (5 \times \text{one}) = 33$

   c. $(1 \times \text{seven} \times \text{seven}) = 49$

   d. $(5 \times \text{seven} \times \text{seven}) + (2 \times \text{seven}) + (4 \times \text{one}) = 263$

4. a. $10_{seven}$

   b. $11_{seven}$

c. $55_{\text{seven}}$

d. $163_{\text{seven}}$

e. $1000_{\text{seven}}$

f. $1010_{\text{seven}}$

5. a. $560_{\text{seven}}$ The 6 means 6 sevens

b. $56_{\text{seven}}$ The 6 means 6 ones

c. $605_{\text{seven}}$ The 6 means 6 (seven × seven)'s or 6 (forty-nines)

d. $6050_{\text{seven}}$ The 6 means 6 (seven × seven × seven)'s or 6 (three hundred forty-three)

6. $\text{seven}^4$ or seven to the fourth power

7. The product of 9 sevens ($7^9$)

8. $205_{\text{ten}}$

9. Even. $13_{\text{seven}}$ and $15_{\text{seven}}$.

10. $132_{\text{seven}}$

11. $100_{\text{seven}}$ or $49_{\text{ten}}$ times as much

12. Yes. Accept any reasonable explanation. $(1 \times 49) + (1 \times 7) + (0 \times 1)$ is divisible by 7.

13. They use seven symbols and seem to have a place value system or base seven. They appear to use |, ∠, △, □, ⧅, ⊠, for 1, 2, 3, 4, 5, 6 and — for zero. ∠∠ follows ∠|.

## 2-5. Computation in Base Seven

Be sure that pupils understand the construction of the addition, base ten, table,

Addition, Base Ten

| + | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 2 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 3 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 4 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| 5 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| 6 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 7 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| 8 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
| 9 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |

Pupils should be helped to observe the symmetry of the table with respect to the diagonal. They will notice that 8 + 6 = 6 + 8, for example and that this is true for any pair of numbers. Later they will learn that this is the commutative property of addition. It is suggested that the word commutative not be used at this time.

If pupils know the facts, no time should be wasted on the table after its characteristics have been discussed.

Addition, Base Seven

| + | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 10 |
| 2 | 2 | 3 | 4 | 5 | 6 | 10 | 11 |
| 3 | 3 | 4 | 5 | 6 | 10 | 11 | 12 |
| 4 | 4 | 5 | 6 | 10 | 11 | 12 | 13 |
| 5 | 5 | 6 | 10 | 11 | 12 | 13 | 14 |
| 6 | 6 | 10 | 11 | 12 | 13 | 14 | 15 |

There is no value in memorizing this table. The process is more important than the facts. The point to be emphasized is that numbers and number properties are independent of the numerals or symbols used to represent the numbers. Commutativity holds in base seven as well as base ten because it is a property of numbers, not numerals.

Addition in base seven is undertaken to clarify addition in decimal notation. Some of the newer elementary school textbooks prefer to use the word "change" or "regroup" rather than "borrow" since the first two words seem to describe the actual process better than the last.

Point out to the pupils that in working in base ten it is often necessary to regroup ten ones as one ten, whereas in base seven we regroup seven ones as one seven.

As pupils use the table in subtraction, they will observe that subtraction is the inverse of addition.

Some of the exercises in addition and subtraction are written in horizontal fashion in preparation for future work in algebra.

### Answers to Exercises 2-5-a

1. a. $56_{seven}$ $(19 + 22 = 41)$

   b. $110_{seven}$ $(41 + 15 = 56)$

   c. $300_{seven}$ $(109 + 38 = 147)$

   d. $620_{seven}$ $(91 + 217 = 308)$

   e. $241_{seven}$ $(33 + 94 = 127)$

   f. $1266_{seven}$ $(199 + 290 = 489)$

   g. $1533_{seven}$ $(327 + 299 = 626)$

2. a. 2 $(7 - 5 = 2)$

   b. 36 $(47 - 20 = 27)$

   c. $163_{seven}$ $(98 - 4 = 94)$

   d. $151_{seven}$ $(91 - 6 = 85)$

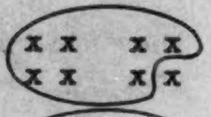
   e. 6 $(32 - 26 = 6)$

   f. $506_{seven}$ $(323 - 72 = 251)$

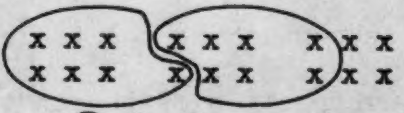
   g. $203_{seven}$ $(247 - 146 = 101)$

   h. $406_{seven}$ $(1715 - 1513 = 202)$

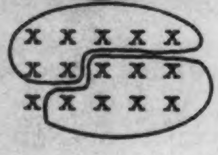
3. a.



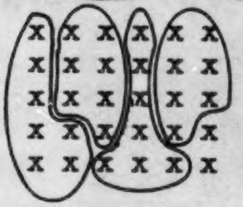
b.



c.



d.



4. Yes

5. (The product of zero and any number is 0. The product of 2 non-zero numbers does not end in zero in base seven numerals)

Multiplication, Base Ten

| × | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 2 | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| 3 | 0 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 |
| 4 | 0 | 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 |
| 5 | 0 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 |
| 6 | 0 | 6 | 12 | 18 | 24 | 30 | 36 | 42 | 48 | 54 |
| 7 | 0 | 7 | 14 | 21 | 28 | 35 | 42 | 49 | 56 | 63 |
| 8 | 0 | 8 | 16 | 24 | 32 | 40 | 48 | 56 | 64 | 72 |
| 9 | 0 | 9 | 18 | 27 | 36 | 45 | 54 | 63 | 72 | 81 |

Study of this table should emphasize the following:

a. The product of 0 and any number is zero.

b. The product of 1 and any number is the number itself.

c. The order in multiplication does not affect the product.

d. The products marked by the diagonal line are squares.

e. The successive products in any one row or column may be found by adding the same number one more time.

Multiplication, Base Seven

| × | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 0 | 2 | 4 | 6 | 11 | 13 | 15 |
| 3 | 0 | 3 | 6 | 12 | 15 | 21 | 24 |
| 4 | 0 | 4 | 11 | 15 | 22 | 26 | 33 |
| 5 | 0 | 5 | 13 | 21 | 26 | 34 | 42 |
| 6 | 0 | 6 | 15 | 24 | 33 | 42 | 51 |

Study of this table is valuable for the additional insight it affords into the understanding of multiplication. There is no value in memorizing it. The table may be used to emphasize that division is the inverse of multiplication.

Since division is the most demanding operation, it is suggested that teachers regard the topic as optional and do only as much as they judge appropriate, in class discussion. Pupils may need help in learning how to use the multiplication table to find division facts. Exercises are included for those pupils who wish to attempt them.

Working with base ten and base seven numerals, using the tables, and changing from base seven to base ten, provide many opportunities for needed drill in the four operations. Teachers may wish to observe the kinds of drill needed and to devise additional exercises of the appropriate type.

### Answers to Exercises 2-5-b

1. a. $45_{seven}$ b. $222_{seven}$ c. $1116_{seven}$ d. $3325_{seven}$

2. a. 5 b. $62_{seven}$ c. $421_{seven}$ with a remainder of 2.

   d. $123_{seven}$ with a remainder of $12_{seven}$.

3. a. $(4 \times 7 \times 7) + (0 \times 7) + (3 \times 1) = 199_{ten}$

   b. $(1 \times 10 \times 10) + (8 \times 10) + (9 \times 1) = 189_{ten}$

4. $403_{seven}$

5. a. $66_{seven}$ b. $123_{seven}$ c. $1061_{seven}$ d. $1205_{seven}$

6. a. $4_{seven}$ b. $26_{seven}$ c. $334_{seven}$

7. Room 123; book 7; 15 chapters; 394 pages; 32 pupils; 5 days; 55 minutes; 13 girls; 19 boys; 11 years old; 5 feet 6 inches tall.

## 2-6. Changing from Base Ten to Base Seven

Pupils in general find it easier to change from base seven to base ten numerals than the reverse. This section gives much detail in order to provide help to pupils who may have trouble.

Ask for the highest power of seven which is contained in the number given in base ten numeration. For example, consider $634_{10}$. Is $7^4$ or $(2401_{10})$ contained in $634_{10}$? Is $7^3$ or $(343_{10})$? After we have taken as many $343_{10}$'s as possible from $634_{10}$, how much remains? The next power to 7 is $7^2$. How many $49_{10}$'s are contained in $291_{10}$? Finally, how many 7's, and how many 1's are left?

A second method of changing from base ten to base seven numerals is developed in Exercises 3, 4, 5 on Page 52.

| | | | |
|---|---|---|---|
| 10 | $\underline{\lfloor 1958}$ | | |
| 10 | $\underline{\lfloor 195}$ | remainder 8 | |
| 10 | $\underline{\lfloor 19}$ | remainder 5 | The number is 1958 |
| 10 | $\underline{\lfloor 1}$ | remainder 9 | |
| | 0 | remainder 1 | |

| | | | |
|---|---|---|---|
| 7 | $\underline{\lfloor 524}$ | | |
| 7 | $\underline{\lfloor 74}$ | remainder 6 | |
| 7 | $\underline{\lfloor 10}$ | remainder 4 | The number is 1346 base seven |
| 7 | $\underline{\lfloor 1}$ | remainder 3 | |
| | 0 | remainder 1 | |

Some pupils will see that the 6 ones are found first when 74 groups of 7 are taken away. These groups of 7 are then put together in groups of 7 sevens with 4 groups left for the numeral in the 7 place. The process is repeated to find the digits for successive places in the numeral.

## Answers to Exercises 2-6

1. a. $50_{\text{ten}} = (1 \times \text{seven}^2) + (0 \times \text{seven}) + (1 \times \text{one}) = 101_{\text{seven}}$

   b. $145_{\text{ten}} = (2 \times \text{seven}^2) + (6 \times \text{seven}) + (5 \times \text{one}) = 265_{\text{seven}}$

   c. $1024_{\text{ten}} = (2 \times \text{seven}^3) + (6 \times \text{seven}^2) + (6 \times \text{seven}) + (2 \times \text{one}) = 2662_{\text{seven}}$

2. a. $15_{\text{seven}}$

   b. $51_{\text{seven}}$

   c. $62_{\text{seven}}$

   d. $104_{\text{seven}}$

   e. $431_{\text{seven}}$

   f. $3564_{\text{seven}}$

3. Q = 195 R = 8

   Q = 19 R = 5

   Q = 1 R = 9

   Q = 0 R = 1

4. Q = 74 R = 6

   Q = 10 R = 4

   Q = 1 R = 3

   Q = 0 R = 1

5. Divide by seven and continue to divide each quotient by seven.

The digits in the numeral sought will be the remainders in order with the first remainder in the "one" place.

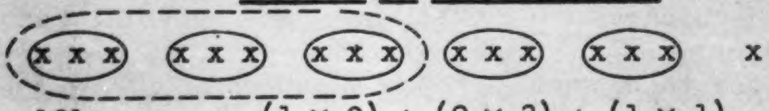
6. a. $1161_{\text{ten}}$
   b. $275_{\text{ten}}$
   c. $654_{\text{seven}}$
   d. $462_{\text{seven}}$
   e. $1116_{\text{seven}}$
   f. 3
   g. $462_{\text{seven}}$

## 2-7. Numerals in Other Bases

Bring out the idea that the base of the system that we use is "ten" for historical rather than mathematical reasons. Some mathematicians have suggested that a prime number such as 7 has certain advantages. The Duodecimal Society of America, 20 Carlton Place, Staten Island 4, New York supports the adoption of twelve as the best number base. Information about the duodecimal system is furnished by this society on request. Exercises in other number bases help establish an understanding of what a positional, power system of numeration is.

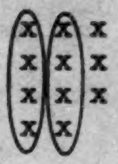
### Answers to Exercises 2-7

1.

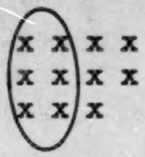


$121_{\text{three}} = (1 \times 9) + (2 \times 3) + (1 \times 1)$

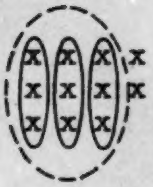
2. a.



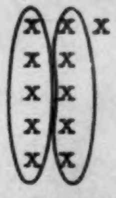
b.



c.



d.



3.

| Base Ten | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Base Five | 0 | 1 | 2 | 3 | 4 | 10 | 11 | 12 | 13 | 14 | 20 |

| Base Ten |  | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Base Five |  | 21 | 22 | 23 | 24 | 30 | 31 | 32 | 33 | 34 | 40 |

| Base Ten |  | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Base Five |  | 41 | 42 | 43 | 44 | 100 | 101 | 102 | 103 | 104 | 110 |

4. Yes

5. a. two b. two c. two d. two

6. a. $(2 \times 36) + (4 \times 6) + (5 \times 1) = 101$

b. $(4 \times 25) + (1 \times 5) + (2 \times 1) = 107$

c. $(1 \times 27) + (0 \times 9) + (0 \times 3) + (2 \times 1) = 29$

d. $(1 \times 64) + (0 \times 16) + (2 \times 4) + (1 \times 1) = 73$

Other answers are acceptable, i.e., $(2 \times 6^2) + (4 \times 6^1)$ , $(5 \times 1)$.

7.

|  | Base Ten | Base Six | Base Five | Base Four | Base Three |
|---|---|---|---|---|---|
| a. | 11 | 15 | 21 | 23 | 32 |
| b. | 15 | 23 | 30 | 33 | 120 |
| c. | 28 | 44 | 103 | 130 | 1001 |
| d. | 36 | 100 | 121 | 210 | 1100 |

8. Two. The binary base. With only one symbol it would be impossible to express both zero and one.

*9. a. $1003_{four}$ e. $111_{four}$

b. $1110_{six}$
f. $1045_{six}$

c. $1002_{three}$
g. $112_{six}$

d. $424_{five}$
h. $22_{three}$

10. The new system is in base four.

| Base ten | New Base | New Base Names |
|---|---|---|
| 0 | 0 | do |
| 1 | \| | ray |
| 2 | ∧ | me |
| 3 | ≥ | fa |
| 4 | \|0 | ray do |
| 5 | \|\| | ray ray |
| 6 | \|∧ | ray me |
| 7 | \|≥ | ray fa |
| 8 | ∧0 | me do |
| 9 | ∧\| | me ray |
| 10 | ∧∧ | me me |

| Base Ten | New Base | New Base Names |
|---|---|---|
| 11 | ∧≥ | me fa |
| 12 | ≥0 | fa do |
| 13 | ≥\| | fa ray |
| 14 | ≥∧ | fa me |
| 15 | ≥≥ | fa fa |
| 16 | \|00 | ray do do |
| 17 | \|0\| | ray do ray |
| 18 | \|0∧ | ray do me |
| 19 | \|0≥ | ray do fa |
| 20 | \|\|0 | ray ray do |

11.

| + | 0 | \| | ∧ | ≥ |
|---|---|---|---|---|
| 0 | 0 | \| | ∧ | ≥ |
| \| | \| | ∧ | ≥ | \|0 |
| ∧ | ∧ | ≥ | \|0 | \|\| |
| ≥ | ≥ | 0 | \|\| | \|∧ |

| × | 0 | \| | ∧ | ≥ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| \| | 0 | \| | ∧ | ≥ |
| ∧ | 0 | ∧ | \|0 | \|∧ |
| ≥ | 0 | ≥ | \|∧ | ∧\| |

## 2-8. The Binary and Duodecimal Systems

The use of binary notation in high speed computers is well known. The binary system is used for computers since there are only two digits, and an electric mechanism is either "on" or "off". Such an arrangement is called a flip-flop mechanism. A number of pamphlets distributed by IBM, Remington Rand, and similar sources may be obtained by request and used for supplementary reading and study. "Yes No - One Zero" published by Esso Standard Oil Co., 15 West 51st., New York 19, New York is available for the asking only in states served by Esso.

It should be of interest that the sum 11001 + 110 looks the same in the binary system, decimal system, and, in fact, all positional number systems. The meaning, however, is quite different.

The base two has the disadvantage that, while only two different digits are used, many more digits are needed to express numbers in binary notation than in decimal, e.g.,

$$2000_{ten} = 11,111,010,000_{two}$$

Pupils may be interested in the remainder method for changing a number from one base to another. This method of changing $25_{ten}$ to binary notation process rests on repeated division by 2, to identify the powers of 2 whose sum is 25. To change $25_{ten}$ to base three, repeat division by 3; to base four, divide by 4; and so on.

The division is shown below for changing $25_{ten}$ to binary notation, followed by an interpretation of the results of the division at each stage. It will be noted that the remainders in reverse order indicate the digits in binary numerals. Recall that $2^0 = 1$.

$$\begin{array}{lll}
2\,)\underline{25} & & 25 = 25 \times 2^0 \\
2\,)\underline{12} & ,\ \mathrm{R}\ 1 & 25 = 12 \times 2^1 + 1 \times 2^0 \\
2\,)\underline{6} & ,\ \mathrm{R}\ 0 & 25 = 6 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 \\
2\,)\underline{3} & ,\ \mathrm{R}\ 0 & 25 = 3 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 \\
2\,)\underline{1} & ,\ \mathrm{R}\ 1 & 25 = 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 \\
2\,)\ 0 & ,\ \mathrm{R}\ 1 & 25 = 0 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0
\end{array}$$

Here is a set of cards which can be used in a number trick.

| 1 | | | |
|---|---|---|---|
| 1 | 9 | 17 | 25 |
| 3 | 11 | 19 | 27 |
| 5 | 13 | 21 | 29 |
| 7 | 15 | 23 | 31 |

| 2 | | | |
|---|---|---|---|
| 2 | 10 | 18 | 26 |
| 3 | 11 | 19 | 27 |
| 6 | 14 | 22 | 30 |
| 7 | 15 | 23 | 31 |

| 4 | | | |
|---|---|---|---|
| 4 | 12 | 20 | 28 |
| 5 | 13 | 21 | 29 |
| 6 | 14 | 22 | 30 |
| 7 | 15 | 23 | 31 |

| 8 | | | |
|---|---|---|---|
| 8 | 12 | 24 | 28 |
| 9 | 13 | 25 | 29 |
| 10 | 14 | 26 | 30 |
| 11 | 15 | 27 | 31 |

| 16 | | | |
|---|---|---|---|
| 16 | 20 | 24 | 28 |
| 17 | 21 | 25 | 29 |
| 18 | 22 | 26 | 30 |
| 19 | 23 | 27 | 31 |

Using the first four cards, tell a person to choose a number between 1 and 15, to pick out the cards containing that number and to give them to you. By adding the numbers at the top of the cards he gives you, you can tell him the number he chose. Note that the numerals at the top of the cards represent the powers of two in reverse order.

By using all five cards, you can pick out numbers from 1 to 31. The trick is based on the application of the binary numbers. For further information, see the following references in the bibliography:

No. 8, pp. 5-6; No. 10, pp. 26-28; No. 13, p. 111.

If you have a peg board and some match sticks, you can represent base two numbers on the board. Leave a hole blank for 0 and put in a match stick for one. Represent two numbers on the board, one below the other, and try adding on the board.

The twelve system uses two digits more than the decimal system. From some points of view twelve is a better choice for a base than ten. Many products are packaged and sold by the dozen and by the gross. Twelve is divisible by 2, 3, 4, and 6 as well as 12. Ten is divisible only by 2, 5, and 10. Because it employs a larger base, large numbers may be represented in base twelve with fewer digits than smaller bases require. For example:

$$\text{TOE}_{\text{twelve}} = 1451_{\text{ten}}$$

## Answers to Exercises 2-8

| Base ten | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Base two | 0 | 1 | 10 | 11 | 100 | 101 | 110 | 111 | 1000 | 1001 | 1010 |

| 11 | 12 | 13 | 14 | 15 | 16 | 18 | 19 | 20 | 21 |
|---|---|---|---|---|---|---|---|---|---|
| 1011 | 1100 | 1101 | 1110 | 1111 | 10000 | 10010 | 10011 | 10100 | 10101 |

| 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
|---|---|---|---|---|---|---|---|---|
| 10110 | 10111 | 11000 | 11001 | 11010 | 11011 | 11100 | 11101 | 11110 |

| 31 | 32 | 33 |
|---|---|---|
| 11111 | 100000 | 100001 |

2. Addition, Base Two

| + | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 10 |

There are only four addition facts.

3. Multiplication, Base Two

| × | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

There are only four multiplication facts. The two tables are not alike, except that $0 + 0$ and $0 \times 0$ both equal 0.

The binary system is very simple because there are only four addition and four multiplication facts to remember. Computation is simple. Writing large numbers, however, is tedious.

4. a. $111_{two} = (1 \times 2^2) + (1 \times 2^1) + (1 \times 1) = 7$

b. $1000_{two} = (1 \times 2^3) + (0 \times 2^2) + (0 \times 2^1) + (0 \times 1) = 8$

c. $10101_{two} = (1 \times 2^4) + (0 \times 2^3) + (1 \times 2^2) + (0 \times 2^1) + (1 \times 1) = 21$

d. $11000_{two} = (1 \times 2^4) + (1 \times 2^3) + (0 \times 2^2) + (0 \times 2^1) + (0 \times 1) = 24$

e. $10100_{two} = (1 \times 2^4) + (0 \times 2^3) + (1 \times 2^2) + (0 \times 2^1) + (0 \times 1) = 20$

5. $22 = 1T_{twelve}$

$23 = 1E_{twelve}$

$24 = 20_{twelve}$

6. a. The letter "T" is the numeral for ten in base twelve

b. 408

7. a. $111_{twelve} = (1 \times 12^2) + (1 \times 12^1) + (1 \times 1) = 157$

b. $3T2_{twelve} = (3 \times 12^2) + (10 \times 12^1) + (2 \times 1) = 554$

c. $47E_{twelve} = (4 \times 12^2) + (7 \times 12) + (11 \times 1) = 671$

d. $TOE_{twelve} = (10 \times 12^2) + (0 \times 12) + (11 \times 1) = 1451$

"$twelve^2$" may be used instead of $12^2$, etc.

8. a. $111_{two}$ b. $1011_{two}$ c. $110000_{two}$ d. $110110_{two}$

(The word "two" may be omitted in answers for 8 and 9 as in the exercise.)

9. a. $10_{two}$ b. $11_{two}$ c. 111 d. 11

10. a. $100011_{two}$ b. $10000000_{two}$ c. $1100_{two}$

d. $1100100_{two}$

*11. a. Addition $323_{twelve}$ b. $1005_{twelve}$

c. Subtraction $149_{twelve}$ d. $85E_{twelve}$

*12. The binary system is extremely simple in computation. Large numbers are tedious to write.

The duodecimal system may be used conveniently to represent large numbers. Twelve is divisible by 2, 3, 4, 6 and 12 while ten is divisible only by 2, 5, and 10. The twelve system requires more computational facts which might increase difficulties in memorizing tables of addition and multiplication. We do use twelve in counting dozens, gross, etc., and in the common measures of length.

*13. a. $2E5_{\text{twelve}}$

b. $378_{\text{twelve}}$

14. Five weights; 1 oz., 2 oz., 4 oz., 8 oz., may be used to check any weight up to 15 ounces. By adding a 16 oz. weight, any weight up to 31 ounces may be checked.

15. People who work with computers often use the base eight. To change from binary to octal and back is simple with the help of the table:

| Binary | Octal |
|---|---|
| 000 | 0 |
| 001 | 1 |
| 010 | 2 |
| 011 | 3 |
| 100 | 4 |
| 101 | 5 |
| 110 | 6 |
| 111 | 7 |

For example, we have

$$2000_{\text{ten}} = 011,111,010,000_{\text{two}} = 3720_{\text{eight}}$$

Note the grouping of numerals by threes in the binary numeral. The sum of the place values of digits in each group results in the octal numeral. Hence

$$011 = (1 \times 2) + (1 \times 1) = 3$$

$$111 = (1 \times 4) + (1 \times 2) + (1 \times 1) = 7, \text{ etc.}$$

## 2-9. Divisibility

The purpose of this section is to compare evidences of divisibility in one notation with evidences in another notation. Pupils should understand what even numbers are, and how to distinguish numerals representing even numbers in base ten and other bases. Having done this, they then can gain some understanding of the properties of numbers as opposed to properties of the numerals. Numerals ending in 0, 2, 4, 6, and 8 represent even numbers in some bases but not in others.

This section is intended to be only a brief introduction to the study of divisibility. Additional material will be introduced in Chapter 5.

### Answers to Exercises 2-9

1. b and d
2. d
3. c and d
4. $0 + 9 = 9$

   $1 + 8 = 9$

   $2 + 7 = 9$

   $3 + 6 = 9$

   $4 + 5 = 9$

   $1 + 8 = 9$

   $1 + 1 + 7 = 9$

   $1 + 8 = 9$

   18 is divisible by 9

   117 is divisible by 9

   Rule: If the sum of the digits is divisible by nine, the number is divisible by nine.

If the sum of the digits is not divisible by nine, the number is not divisible by nine. (Note: This applies to our base ten numerals.)

5. Rule: In base ten notation, if the sum of the digits is divisible by 3, the number is divisible by 3. If the sum of the digits is not divisible by 3, the number is not divisible by 3.

6. Rule: In base ten notation, a number is divisible by four if and only if the last two digits in the numeral representing the number represent a number divisible by four. For example: $12\underline{36}$, $39\underline{12}$, and $1234\underline{56}$ are divisible by four.

7. Rule: If the sum of the digits of a number in base seven notation is divisible by six, the number is divisible by six.

   Thus: in $15_{seven}$, $1 + 5 = 6$. Hence $15_{seven}$ is divisible by 6; in $33_{seven}$, $3 + 3 = 6$. Hence $33_{seven}$ is divisible by six.

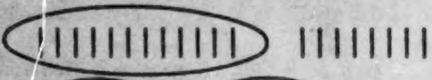
8. The rule for divisibility by 3 in base ten notation holds for base seven. (See answer for exercise 5).

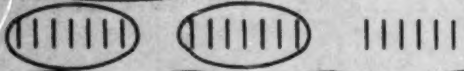
   The rule will work for all bases which may be expressed as $10 \pm 3n$ where n is any integer. Hence, the rule holds when $n = 1$, or bases seven and thirteen; $n = 2$, or bases four and sixteen, etc.

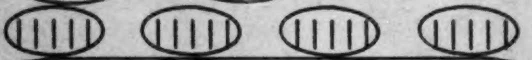
## 2-10. Summary

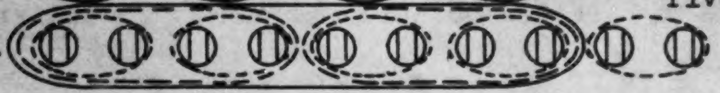
The discussion is self-explanatory. Expansion on any of the topics is left to the discretion of the teacher.

### Answers to Exercises 2-10

1. a.  = $18_{twelve}$

b.  = $26_{seven}$

c.  = $40_{five}$

d.  = $10100_{two}$

2. a. twelve b. seven c. five d. two

3. a. $111_{two} = (1 \times 2^2) + (1 \times 2) + (1 \times 1) = 7$

b. $321_{four} = (3 \times 4^2) + (2 \times 4) + (1 \times 1) = 57$

c. $402_{five} = (4 \times 5^2) + (0 \times 5) + (2 \times 1) = 102$

d. $2631_{seven} = (2 \times 7^3) + (6 \times 7^2) + (3 \times 7) + (1 \times 1) = 1002$

e. $37T_{twelve} = (3 \times 12^2) + (7 \times 12) + (10 \times 1) = 526$

4. a. $6E4_{twelve}$ b. $1750_{eight}$ c. $13{,}000_{five}$

d. $1{,}101{,}001_{three}$ e. $1{,}111{,}101{,}000_{two}$

5. $100_{nine}$ $101_{nine}$ $102_{nine}$ $103_{nine}$ $104_{nine}$

6\.

| Base Two | | Base Eight | | Base Ten |
|---|---|---|---|---|
| 1011 | = | 13 | = | 11 |
| 101 | = | 5 | = | 5 |
| $10000_{two}$ | = | $20_{eight}$ | = | $16_{ten}$ |

7. $$\begin{array}{rcr} \textbf{Base Ten} & & \textbf{Base Six} \\ 210 & = & 550 \\ \underline{123} & = & \underline{323} \\ 87_{ten} & = & 223_{six} \end{array} = (2 \times 36) + (2 \times 6) + (3 \times 1) = 87_{ten}$$

$$\begin{array}{rcr} \textbf{Base ten} & & \textbf{Base Five} \\ 210 & = & 1320 \\ \underline{123} & = & \underline{443} \\ 87_{ten} & = & 322_{five} \end{array} = (3 \times 25) + (2 \times 5) + (2 \times 1) = 87_{ten}$$

8. $18_{eleven} = 19_{ten}$

$19_{eleven} = 20_{ten}$

$1T_{eleven} = 21_{ten}$

$20_{eleven} = 22_{ten}$

9. **Base Twenty**

$47_{20} = 87_{ten}$

10. $DCBAO = 43{,}210_{five}$

$43{,}210_{five} = (4 \times 625) + (3 \times 125) + (2 \times 25) + (1 \times 5) + (0 \times 1) = 2930_{ten}$

11. The base is twenty-six.

Counting Table

| Base Ten | Base Twenty-six | Base Ten | Base Twenty-six | Base Ten | Base Twenty-six |
|---|---|---|---|---|---|
| 0 | O | 9 | I | 18 | S |
| 1 | A | 10 | J | 19 | T |
| 2 | B | 11 | K | 20 | U |
| 3 | C | 12 | L | 21 | V |
| 4 | D | 13 | M | 22 | W |
| 5 | E | 14 | N | 23 | X |
| 6 | F | *15 | P | 24 | Y |
| 7 | G | 16 | Q | 25 | Z |
| 8 | H | 17 | R | | |

*"O" is used as zero

a. $BE_{\text{twenty-six}} = (B \times 26^1) + (E \times 1)$

$= (B \times 26) + (E \times 1)$

$= (2 \times 26) + (5 \times 1)$

$= 57_{\text{ten}}$

b. $TWO_{\text{twenty-six}} = (T \times 26^2) + (W \times 26^1) + (O \times 1)$

$= (T \times 676) + (W \times 26) + (O \times 1)$

$= (19 \times 676) + (22 \times 26) + (0 \times 1)$

$= 12844 + 572 + 0$

$= 13416_{\text{ten}}$

c. $Four_{\text{twenty-six}} = (F \times 26^3) + (O \times 26^2) + (U \times 26^1) +$

$(R \times 1)$

$= (6 \times 17576) + (0 \times 676) + (20 \times 26) +$

$(17 \times 1)$

$= 105{,}456 + 0 + 520 + 17$

$= 105{,}993_{\text{ten}}$

12. The method works for base twelve and base seven. It will also work for other bases. For bases larger than ten, add. For bases less than ten, subtract.

Example in Base Six:

$44_{six}$ multiplying: $(4 \times 4) = 16$

$44 - 16 = 28_{ten}$

Example in Base Fifteen:

$46_{fifteen}$ multiplying: $(4 \times 5) = 20$

$46 + 20 = 66_{ten}$

Students may suggest other methods which should be checked carefully for validity.

## Sample Test Questions for Chapter 2

### Part I. True - False

1. The 3 in $356_{seven}$ stands for three hundred.
2. $10^4$ means 10 × 10 × 10 × 10.
3. The numeral 8 means the same numbers in the ten system and the twelve system.
4. The smaller the base, the more basic combinations there are in the multiplication table.
5. The fourth place from the right in the decimal system has the place value $10^5$.
6. In base two numerals the number after 100 is 1000.
7. We can make a symbol to mean what we wish.
8. When we "carry" in addition the value of what is carried depends upon the base.
9. A number may be written in numerals with any whole number greater than one as a base.
10. In the symbol $6^3$, the exponent is 3.
11. $513_{six}$ means (5 × six × six × six) + 1 × six × six) + (3 × six)
12. The 1 in 10,000 (base two) means $1 \times 2^4$ or sixteen.
13. The following numerals represent the same number: $183_{twelve}$; $363_{eight}$; $1033_{six}$.
14. In base eight numerals, the number before 70 is 66.
15. Four symbols are sufficient for a numeration system with

base five.

16. In the base four system $3 + 3 = 11_{\text{four}}$

17. When we "borrow" in the twelve system as in 157 - 6E, we actually "borrow" 12 units.

18. In the Egyptian system a single symbol could be used to represent a collection of several things.

19. The Babylonians made use of place value in their numeration system.

20. The Roman numeral system had a symbol for zero.

Part II. Completion

1. In decimal numerals $14_{\text{twelve}}$ is ____.
2. MCXXIV in decimal numerals is ____.
3. The decimal system uses ____ different symbols.
4. In any numeration system, the smallest place value is ____.
5. 629,468,000,000 written in words is ____________________ . ______________________________.
6. The number represented by $212_{\text{seven}}$ is ______ (even or odd).
7. In expanded notation $5{,}678_{\text{ten}}$ is _________.
8. $625_{\text{seven}} + 344_{\text{seven}} = \_\_\_\_\_\_\_\_\_\_\_\__{\text{seven}}$
9. Multiply $312_{\text{four}}$ by $32_{\text{four}}$.
10. $110011_{\text{two}} = \_\_\_\_\_\_$ ten
11. The numeral $444_{\text{five}}$ represents an ____ (even, odd)

12. Add: $42_{five} + 14_{five}$ = ________.

13. $13_{ten}$ = __________ $_{two}$.

14. The numeral after $37_{eight}$ is __________ $_{eight}$.

15. Using the digits 5, 6, 7, and 0, write the largest possible 4-place numeral. ______.

16. Using the digits, 5, 6, 7, and 0, write the smallest possible 4-place numeral. ______

17. What is the largest number you can write, using only two 4's?_____

18. Write this numeral without exponents: $5^3$ _____

19. The numeral immediately before $1000_{two}$ is _____.

20. Subtract: $42_{five} - 14_{five}$ = __________

## Part III. Multiple-Choice

1. In which of the numerals below does 1 stand for four?

   a. $21_{four}$

   b. $21_{eight}$

   c. $100_{two}$

   d. $102_{three}$

   e. None of the above is correct.

2. In what base are the numerals written if 2 × 2 = 10?

   a. Base two

   b. Base three

   c. Base four

d. Base five

e. All of the above are correct.

3. A decimal numeral which represents an odd number is:

a. 461,000

b. 7629

c. 5634

d. 9,000,000

e. None of the above is correct.

4. If N represents an even number, the next consecutive even number can be represented by:

a. N

b. N + 1

c. N + 2

d. 2N

e. All of the above are correct.

5. Which numeral represents the largest number?

a. $43_{\text{five}}$

b. $212_{\text{three}}$

c. $10110_{\text{two}}$

d. $24_{\text{nine}}$

e. $10_{\text{twenty-five}}$

6. Which is correct?

a. $5^4 = 5 + 5 + 5 + 5$

b. $4^3 = 4 \times 4 \times 4$

c. $5^4 = 4 \times 4 \times 4 \times 4 \times 4$

d. $2^3 = 2 \times 3$

e. None of the above is correct.

7. $612_{nine}$ is how many times as large as $6120_{nine}$?

a. twelve

b. ten

c. nine

d. five

e. None of the above is correct.

8. In which base does the numeral 53 represent an even number?

a. twelve

b. ten

c. eight

d. seven

e. six

## Answers to Sample Test Questions for Chapter 2

| Part I<br>True - False | Part II<br>Completion | Part III<br>Multiple Choice |
|---|---|---|
| 1. False | 1. 16 | 1. c |
| 2. True | 2. 1124 | 2. c |
| 3. True | 3. Ten | 3. b |
| 4. False | 4. One | 4. c |
| 5. False | 5. Six hundred twenty-nine billion, four hundred sixty-eight million | 5. e |
| 6. False | 6. Odd | 6. b |
| 7. True | 7. $(5 \times 10^3) + (6 \times 10^2) + (7 \times 10^1) + (8 \times 1)$ | 7. c |
| 8. True | 8. $1302_{\text{seven}}$ | 8. d |
| 9. True | 9. $23310_{\text{four}}$ | |
| 10. True | 10. 51 | |
| 11. False | 11. even | |
| 12. True | 12. $111_{\text{five}}$ | |
| 13. False | 13. $1101_{\text{two}}$ | |
| 14. False | 14. $40_{\text{eight}}$ | |
| 15. False | 15. 7650 | |
| 16. False | 16. 5067 | |
| 17. True | 17. 44 | |
| 18. True | 18. 125 | |
| 19. True | 19. $111_{\text{two}}$ | |
| 20. False | 20. $23_{\text{five}}$ | |

## Glossary

BASE: 1. In a power, the number which is to be taken as factor. (See POWER).

2. In a system of numeration using place value, the number whose powers are associated with the successive places in a numeral. For example, if five is the base of the system, then in any numeral the first digit on the right indicates the number of 1's, the second (from the right) the number of 5's, the third the number of 5x5's, and so on. Thus, if 5 is the base, the numeral 1234 is to be understood as: (4x1) + (3x5) + (2x5x5) + (1x5x5x5).

BINARY NOTATION: To write a number in binary notation is to represent the number in base two numerals.

DECIMAL NOTATION: To write a number in decimal notation is to represent the number in base ten numerals.

DIGIT: The name given to any one of the single symbols which are used in writing a numeral in a system based on place value. In the decimal system, there are ten digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. In base seven notation there are seven digits: 0, 1, 2, 3, 4, 5, 6.

DUODECIMAL NOTATION: To write a number in duodecimal notation is to represent the number in base twelve numerals.

EXPANDED NOTATION: An expression for a number as the sum of multiples of 1 and of powers of a specified base.
Examples: $536_{\text{ten}} = (5 \times 10^2) + (3 \times 10^1) + (6 \times 1)$
$536_{\text{seven}} = (5 \times \text{seven}^2) + (3 \times \text{seven}^1) + (6 \times 1)$

EXPONENT: A number symbol written at the right and above another number symbol to show how many times the latter is to be taken as factor. (See POWER)

FACTOR: In multiplication, any one of the numbers that form a product.
Examples: In (8 × 5), 8 is a factor and 5 is a factor.
In (7 × 3 × 12), 7 is a factor, 3 is a factor, and 12 is a factor.

NOTATION: Any systematic convention for expressing ideas, relationships, and the like.

**NUMERAL:** A symbol used to represent a number.

**NUMERATION:** Any system for naming or writing numbers.

**OCTAL NOTATION:** To write a number in octal notation is to represent the number in base eight numerals.

**PLACE VALUE:** The value assigned to a place, and by which the digit in that place is multiplied, determined by its position in reference to other digits in the entire numeral.

**POWER:** For a natural number b, the product $(a \times a \times a \ldots \times a)$, written $a^b$, where a appears as factor b times. For example, $5^3 = 5 \times 5 \times 5$. Similarly, $4^2 = 4 \times 4$, and $3^1 = 3$. In such expressions, the number which occurs as factor is the base and the number of times it occurs is the exponent. Thus in the expression $5^3$, 5 is the base and 3 is the exponent.

**SYMBOL:** A mark of any sort which represents a number, relationship, operation, and the like.

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# CHAPTER 3

# WHOLE NUMBERS

## 3-1. Counting Numbers

Understandings to be developed in this section:

(a) Number is an idea.

  (1) Small numbers can be learned without counting, some large ones by patterns. (Cards, Dominoes)

  (2) The number of members of any set can be found by matching the members of the set with some standard set. This is a clumsy method if the number of members is large, as the standard sets must themselves be large.

  (3) Matching the members of a set with a memorized set of sounds representing the counting numbers is the best known way of finding the number of members in a set.

(b) One-to-one correspondence.

  Two sets are in a one-to-one correspondence when each member of one set is matched with only one member of the other set and vice-versa.

(c) "Counting Numbers" are sometimes called "natural numbers".

(d) The name that we give to the set of counting numbers and zero is "whole numbers". The counting numbers do not include zero.

## Answers to Exercises 3-1

1. (a) 1, 2, 3, 4, 5, 6.

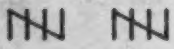
   (b) 1 + 1, 2 + 1, 3 + 1, 0 + 5, 5 + 1, 1 + 6, 5 + 3.

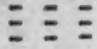
   (c) IV, V, VI, VII, VIII, IX, X, XI.

   (d) $1_{seven}$, $2_{seven}$, $3_{seven}$, $4_{seven}$, $5_{seven}$, $6_{seven}$, $10_{seven}$

2. none; 2 and 5

3. Putting them in a 5 × 5 square will probably be as simple a pattern as any other.

4. (a) 

   (b) 

5. Multiply 4 by 3; (2 × 5) + (2 × 1); 2 × 6 on each side of diagonal; 2 × 6 on each side of the vertical line segment; any other way that doesn't use one-to-one correspondence.

6. French - un; Spanish - unos; German - ein; Russian - adeen

7. IV corresponds to two dots.

8. 8 was left out.

9. 17

10. He subtracted 27 from 81 - No, the answer should be 55.

11.

| Two-to-one correspondence | Four-to-one correspondence |
|---|---|
| (a) Feet to people | (a) Tires to cars, wagons |
| (b) Ears, eyes, arms, hands, etc. to people. | (b) Legs to dogs, horses, cows |

12. The following illustrates a one-to-one correspondence between the counting numbers and the even numbers.

## 3-2a The Commutative Property

The principal objective in the study of the commutative, associative, and distributive properties is to have the pupils understand their statements in mathematical language; to distinguish one of the properties from another; and to recognize the one, or ones, that may be used in such problems as in Exercises 3 and 4. These are not properties that are being proved. The pupils have used them for a long time, but they probably have not had names for them and have not recognized when they have been using them.

The pupils will be helped in understanding the commutative property if they have some small objects with which to work. Pieces of paper will do but cubical blocks will be better. Cubical blocks will be more useful in illustrating the associative property for multiplication. Emphasize that the commutative property is a property of (belongs to) the operation of addition and is not dependent upon the numeration system that is used. Use Roman Numerals and numbers in base five to illustrate. For example in base five add

$12 + 23 = 23 + 12 = 40$

$$\underbrace{\text{*****}\quad\text{**}}_{12}\qquad\underbrace{\text{*****}\quad\text{*****}\quad\text{***}}_{23}\qquad\underbrace{\text{*****}\quad\text{*****}\quad\text{*****}\quad\text{*****}}_{40}$$

This is a good opportunity to review numbers in numeration systems other than that with base ten.

Students should be taught to add "up the column" as well as "down the column". Use three whole numbers in the addition since the use of three numbers will not only illustrate commutativity but will lead very naturally into associativity.

$$\begin{array}{r} 476 \\ 324 \\ \underline{935} \end{array}\Bigg\downarrow \qquad\qquad \begin{array}{r} \overline{476} \\ 324 \\ 935 \end{array}\Bigg\uparrow$$

## Commutative Property for Multiplication

Diagrams of stars on the blackboard, arrangements of the blocks in rows and columns will be helpful.

***** 5 rows of 4 stars in each row
***** 4 columns of 5 stars in each column
*****
*****

After repeated arrangements of stars (or blocks) in rows and columns strive for understanding of $a \cdot b = b \cdot a$ where $a$ and $b$ represent any whole numbers.

### Answers to Exercises 3-2a

1. Parts a, c, d, f, h, i, m are true
   Parts b, e, g, j, k, l are false
2. (a) 644 (b) 110,596 (c) 155,752 (d) $105_{seven}$
3. (a) $7 + 4 = 4 + 7$ (f) $(3 \cdot 2) + 5 = 5 + (3 \cdot 2)$
   (b) $12 \cdot 5 > 5 \cdot 11$ (g) $8 - 3 < 9 - 3$
   (c) $23 \cdot 12 < 12 \cdot 32$ (h) $86 \cdot 135 = 135 \cdot 86$
   (d) $3 < 6$ (i) $24 + 3 > 3 + 24$
   (e) $16 > 9 > 3$ (j) $b > c$ or $b = c$
4. (a) 2052 (b) 25,620 (c) 289,884 (d) $1135_{seven}$
5. (a) 5 (b) 5 (c) 0 (d) 0, 1 (e) 46
   (f) 0, 1, 2, 3, 4, 5, 6 (g) 0, 1, 2, 3, 4 (h) any whole number

### Answers to Exercises 3-2b

1. Result is unchanged in parts a, b, c
   Result is changed in parts d, e, f, g, h, i
2. The activities are commutative in parts a, c, e
3. Yes

4. The operation in part d is commutative. (The operations in parts a, b, c are commutative only if the first number in each is equal to the second.)

5. Examples of commutative activities.

To wash your face and wash your hair.

To go north one block and then west one block.

To count to 100 and write the alphabet.

Examples of activities which are not commutative.

To put out the cat and go to bed.

To eat dinner and get up from the table.

To rake the leaves and burn them.

## 3-3. The Associative Property

Have the students use blocks or disks to make such arrangements as *** ** ****

Then have them push the first two sets together and count the total $(3 + 2) + 4$. Then after rearranging have them push the second two sets together and count the total $3 + (2 + 4)$. Use sufficient variations of this procedure to lead to the understanding that $(a + b) + c = a + (b + c)$ where $a, b, c$ are any counting numbers.

Then ask: Is the product $(3 \times 4) \times 5$ equal to the product $3 \times (4 \times 5)$?

This may be illustrated by arranging a set of 20 blocks in a rectangular array, 4 by 5. Then put two layers of 20 blocks each on top of these forming a box arrangement. Look at it in different ways to see $(3 \times 4) \times 5$ and $3 \times (4 \times 5)$. Different boxes may be made to illustrate $2 \times (3 \times 4)$, $(2 \times 3) \times 4$ and many others. Again, emphasis is upon arrival at understanding that $a \cdot (b \cdot c) =$

$(a \cdot b) \cdot c$ where $a, b, c$ are any whole numbers.

Point out some operations or activities which are not associative and have students suggest others.

## Exercises 3-3

1. (a) $$\begin{aligned} (4 + 7) + 2 &= 4 + (7 + 2) \\ 11 + 2 &= 4 + 9 \\ 13 &= 13 \end{aligned}$$ Associative property for addition

(b) $$\begin{aligned} 8 + (6 + 3) &= (8 + 6) + 3 \\ 8 + 9 &= 14 + 3 \\ 17 &= 17 \end{aligned}$$ Associative property for addition

(c) $$\begin{aligned} 46 + (73 + 98) &= (46 + 73) + 98 \\ 46 + 171 &= 119 + 98 \\ 217 &= 217 \end{aligned}$$ Associative property for addition

(d) $$\begin{aligned} (6 \cdot 5) \cdot 9 &= 6 \cdot (5 \cdot 9) \\ 30 \cdot 9 &= 6 \cdot 45 \\ 270 &= 270 \end{aligned}$$ Associative property for multiplication

(e) $$\begin{aligned} (21 + 5) + 4 &= 21 + (5 + 4) \\ 26 + 4 &= 21 + 9 \\ 30 &= 30 \end{aligned}$$ Associative property for addition

(f) $$\begin{aligned} (9 \cdot 7) \cdot 8 &= 9 \cdot (7 \cdot 8) \\ 63 \cdot 8 &= 9 \cdot 56 \\ 504 &= 504 \end{aligned}$$ Associative property for multiplication

(g) $$\begin{aligned} 436 + (476 + 1) &= (436 + 476) + 1 \\ 436 + 477 &= 912 + 1 \\ 913 &= 913 \end{aligned}$$ Associative property for addition

(h) $$\begin{aligned} (57 \cdot 80) \cdot 75 &= 57 \cdot (80 \cdot 75) \\ (4560) \cdot 75 &= 57 \cdot 6000 \\ 342{,}000 &= 342{,}000 \end{aligned}$$ Associative property for multiplication

2. (a) No. (b) No. (c) There is no associative property for subtraction, or the associative property for subtraction does not hold.

3. (a) No. (b) No. (c) $(75 \div 15) \div 5 = 1$

(d) $75 \div (15 \div 5) = 25$ (e) $80 \div (20 \div 2) = 8$

(f) $(80 \div 20) \div 2 = 2$

(g) The associative property does not hold for division.

4. (a) Either $(6 + 1) + 9 = 7 + 9 = 16$, or

$(6 + 1) + 9 = 6 + (1 + 9)$ Associative property for addition
$= 6 + 10$
$= 16$

(b) $2\cdot(13\cdot 10) = 2\cdot 130 = 260$ Pupils may prefer this way

(c) $(12\cdot 9)\cdot 10 = (108)\cdot 10 = 1080$

(d) $4\cdot(25\cdot 76) = (4\cdot 25)\cdot 76$ Associative property for multiplication
$= 100\cdot 76$
$= 7600$

(e) $340 + (522 + 60) = 340 + (60 + 522)$ Commutative property for addition
$= (340 + 60) + 522$ Associative property for addition
$= 400 + 522$
$= 922$

(f) $(5\cdot 67)\cdot 2 = (67\cdot 5)\cdot 2$ Commutative property for multiplication
$= 67\cdot(5\cdot 2)$ Associative property for multiplication
$= 67\cdot 10$
$= 670$

## 3-4. The Distributive Property

Emphasize that the use of two operations in this property makes it easy to distinguish it from the two preceding properties.

The blocks can be used in the following way. Lay out 2 rows of 3 each and 2 rows of 5 each.

*** *****
*** *****

Ask: If we move these together, we will have 2 times what? When they move them together do they get 2 times 8? This can be repeated until they understand that

$$a\cdot(b + c) = (a\cdot b) + (a\cdot c)$$

when $a, b, c$ are whole numbers. Repetition of the same illustration with different numbers of blocks may be better than different types of illustrations. For bright students, the distributive property can be extended to 3 terms. This can be illustrated and used with 3 digit numbers. The general form is:

$$a(b + c + d) = ab + ac + ad$$

Use the distributive property to help make mental computations such as:

$$7 \cdot 32 = 7 \cdot (30 + 2) = (7 \cdot 30) + (7 \cdot 2) = 210 + 14 = 224$$

$$35 \cdot 8 = (30 + 5) \cdot 8 = (30 \cdot 8) + (5 \cdot 8) = 240 + 40 = 280$$

$$5 \cdot 152 = 5 \cdot (100 + 50 + 2) = (5 \cdot 100) + (5 \cdot 50) + (5 \cdot 2) = 500 + 250 + 10 = 760$$

## Answers to Exercises 3-4

1. (a) 33 (b) 45 (c) 71 (d) 135 (e) 45 (f) 18
   (g) 21 (h) 72 (i) 60 (j) 160 (k) 160 (l) 108

2. (a) $4 \cdot 12 = 28 + 20$; $48 = 48$
   (b) $18 + 24 = 6 \cdot 7$; $42 = 42$
   (c) $48 + 42 = 15 \cdot 6$; $90 = 90$
   (d) $23 \cdot 5 = 46 + 69$; $115 = 115$
   (e) $11 \cdot 7 = 33 + 44$; $77 = 77$
   (f) $30 + 18 = 6 \cdot 8$; $48 = 48$
   (g) $2 \cdot 24 = 32 + 16$; $48 = 48$
   (h) $12 \cdot 5\frac{1}{4} = 60 + 3$; $63 = 63$
   (i) $3216 + 3484 = 67 \cdot 100$; $6700 = 6700$
   (j) $36 + 36 = 72 \cdot 1$; $72 = 72$

3. (a) $3 \cdot (4 + \underline{3}) = (3 \cdot 4) + (3 \cdot 3)$
   (b) $2 \cdot (\underline{4} + 5) = (2 \cdot 4) + (\underline{2} \cdot 5)$
   (c) $13 \cdot (6 + 4) = (13 \cdot \underline{6}) + (13 \cdot \underline{4})$
   (e) $(\underline{6} \cdot 4) + (\underline{7} \cdot 4) = (6 + 7) \cdot \underline{4}$

4. (a) $9 \cdot (8 + 2)$
   (b) $(8 \cdot 14) + (8 \cdot 17)$
   (c) $(12 \cdot 5) + (12 \cdot 7)$
   (d) $(13 \cdot 6) + (27 \cdot 6)$
   (e) $(15 \cdot 6) + (15 \cdot 13)$
   (f) $12 \cdot (5 + 4)$

5. (a) $(5\cdot 7) + (5\cdot 8) = 5\cdot(7 + 8)$

   (b) $(3\cdot 4) + (3\cdot 5) = 3\cdot(4 + 5)$

   (c) $(5\cdot 11) + (5\cdot 2) + 5\cdot(11 + 2)$

   (d) $(3\cdot 9) + (3\cdot 17) = 3\cdot(9 + 17)$

   (e) $(5\cdot 20) + (5\cdot 23) = 5\cdot(20 + 23)$

   (f) $(3\cdot 10) + (3\cdot 7) = 3\cdot(10 + 7)$

6. The following parts are true: b, c, d.

   The following parts are false: a, e.

7. (a)
$$\begin{aligned}(20 + 7)\cdot(30 + 4) &= 20\cdot(30 + 4) + 7\cdot(30 + 4)\\ &= (20\cdot 30) + (20\cdot 4) + (7\cdot 30) + (7\cdot 4)\\ &= 600 + 80 + 210 + 28\\ &= 918\end{aligned}$$

   (b)
$$\begin{aligned}(10 + 3)\cdot(20 + 2) &= 10\cdot(20 + 2) + 3\cdot(20 + 2)\\ &= (10\cdot 20) + (10\cdot 2) + (3\cdot 20) + (3\cdot 2)\\ &= 200 + 20 + 60 + 6\\ &= 286\end{aligned}$$

   (c)
$$\begin{aligned}(30 + 7)\cdot(30 + 3) &= 30\cdot(30 + 3) + 7\cdot(30 + 3)\\ &= (30\cdot 30) + (30\cdot 3) + (7\cdot 30) + (7\cdot 3)\\ &= 900 + 90 + 210 + 21\\ &= 1221\end{aligned}$$

   (d)
$$\begin{aligned}(60 + 4)\cdot(60 + 6) &= 60\cdot(60 + 6) + 4\cdot(60 + 6)\\ &= (60\cdot 60) + (60\cdot 6) + (4\cdot 60) + (4\cdot 6)\\ &= 3600 + 360 + 240 + 24\\ &= 4224\end{aligned}$$

$$
\begin{aligned}
\text{(e)}\quad (70 + 5)\cdot(70 + 5) &= 70\cdot(70 + 5) + 5\cdot(70 + 5)\\
&= (70\cdot 70) + (70\cdot 5) + (5\cdot 70) + (5\cdot 5)\\
&= 4900 + 350 + 350 + 25\\
&= 5625\\
\text{(f)}\quad (20 + 1)\cdot(20 + 9) &= 20\cdot(20 + 9) + 1\cdot(20 + 9)\\
&= (20\cdot 20) + (20\cdot 9) + (1\cdot 20) + (1\cdot 9)\\
&= 400 + 180 + 20 + 9\\
&= 609
\end{aligned}
$$

8.

| | |
|---|---|
| From (a) to (b) | Commutative property for multiplication |
| From (b) to (c) | Associative property for multiplication |
| From (c) to (d) | Commutative property for addition |
| From (d) to (e) | Associative property for addition |
| From (e) to (f) | Commutative property for multiplication |
| From (f) to (g) | Associative property for multiplication |
| From (g) to (h) | Distributive property |

## 3-3. The Closure Property

Emphasis here is placed upon (1) understanding meaning of set and (2) meaning of a set closed under an operation. Pupils will give many examples of sets. There is just one empty set. The set of marbles which has no marbles in it is the same as the set of coins which a pupil has in his pocket if he has no coins.

Good examples of sets closed under addition:

The set of whole numbers.

The set of counting numbers.

Then ask if they are closed under multiplication. Under subtraction? Under division?

Emphasize that if just one pair of counting numbers can be found such that their difference (or quotient) is not a counting number, then the set of counting numbers is not closed under subtraction (or division). For example, 9-12 is not a counting number for there is no counting number which can be added to 12 to get 9 and 12/9 is not a counting number since there is no counting number which can be multiplied by 9 to get 12.

## Answers to Exercises 3-5a

1. (a) No. The sum of 2 odd numbers is always an even number.
   (b) No.
2. Yes. Since the sum of 2 even numbers is always an even number.
3. Yes. Since each of the numbers in the set is a multiple of 5, the sum of any two numbers in the set is a multiple of 5.
4. Each set is closed under multiplication.
5. (a) Yes.
   (b) No. For example 500 + 501 = 1001 and 1001 is not in the set.
   (c) No. For example 3 + 47 = 50 and 50 is not in the set.
   (d) Yes. If the numerals of 2 numbers end in 0, then the sum of the numbers ends in 0.
6. (a) Yes. (b) No. (c) No. (d) Yes.
7. Yes. Multiplication is an abbreviated process for addition.
8. No.
9. No.

## Answers to Exercises 3-5b

1. (a) 8578 (b) 3165
2. 19997
3. 221312
4. 407
5. $27\frac{3}{4}$
6. (a) 4596 (b) 1470 (c) 5081
7. \$20.10
8. \$389.37
9. 4800
10. Two million seventy thousand three hundred fifty-one.
11. 72 cents
12. (a) 78,528 (b) 450,954 (c) 2,499,574

## 3-6. Inverse Operations

The basic concepts in this section are:

1. The meaning of inverse.
2. Addition and multiplication have the closure, commutative, and associative properties, while their inverses do not.

The meaning of inverse may be explained by giving an example. "I write on the chalkboard" may be stated as one actually writes "inverse" on the board. Then the teacher may say "The inverse of writing on the board is erasing the board." The board may be actually erased. If the phrase "opposite or inverse" is used it may help the pupil understand the meaning of inverse. It should be emphasized that the inverse operation undoes the first operation. Some pupils may think that the failure to do an operation is the inverse of the operation. For example, to the question "What is the inverse of singing?" the pupil may say "Not singing". "Not singing" is the negation of "singing" and is not the inverse of "singing". In this connection it is important to point out that some operations have no inverse.

Some discussion of $a \cdot x = b$ may be helpful to many students. The following questions may be suggestive.

1. What operation is indicated by $a \cdot x$?
2. What operation will undo multiplication?
3. What is the inverse of multiplication?
4. To undo $a \cdot x$, do we divide $a \cdot x$ by $a$ or $a \cdot x$ by $x$? (Since $a \cdot x$ means $a$ times $x$ we divide by $a$, the multiplier.)
5. How do we undo $3 \cdot 2$? $8 \times 4$? Divide 6 by 3; divide 32 by 8?
6. In terms of these symbols, can you define division?

An understanding of $a \cdot x = b$ will be helpful to the pupil as he studies percentage, and the equivalence of the two statements "$b \div a = x$" and "$a \cdot x = b$" will be of great importance in Chapter 6. Therefore, an emphasis on understanding the relationship between a, x, and b is desirable.

## Answers to Exercises 3-6

1. The operations in the following parts have inverses:
   a, b, c, d, e, f, h, j, l, o.
2. (a) Putting down the pencil.
   (b) Take off your hat.
   (c) Getting out of a car.
   (d) Withdraw your hand.
   (e) Divide.
   (f) Tear down.
   (h) Step backward.
   (j) Subtraction.
   (l) Addition.

(o) Putting on a tire.

3. (a) 46462 (b) $507.10 (c) 506 ft. (d) $1412.78
(e) $1101.04 (f) $1342.67 (g) 876 (h) 987
(i) 798 (j) 697 (k) 33 (l) 1476 (m) 68
(n) 143 (o) 58140

4. (a) 5 (b) 5 (c) 1 (d) 7 (e) None (f) 0
(g) 8 (h) 4 (i) 3 (j) 3 (k) None (l) 7
(m) Any whole number (n) 4 (o) 5 (p) 5 (q) 9
(r) 9 (s) 6 (t) 0 (u) 0 (v) 0 (w) 0
(x) 1 (y) 1 (z) 1

5. (a) 19 (b) 1992 (c) 89 (d) 19,219 (e) 165821
(f) 13 (g) 6 (h) 20

6. (a) 21 (b) 84 (c) 202 (d) 3 (e) 46 (f) 20
(g) 104 (h) 5 (i) 11 (j) 11

## 3-7. Betweenness and the Number Line

Understandings

1. The number line helps to show how the counting numbers are related. The students may ask about the dots to the left of zero. These dots will be discussed next year.

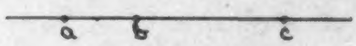
2. A number is less than a second number if the first is to the left of the second. A number is greater than another if it is to the right of it.

3. There is not always a counting number between two counting numbers.

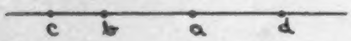
4. To find the number of whole numbers between two other numbers (if it can be done at all): Subtract the smaller from the larger and then subtract one (1) from the remainder. Or, subtract one (1) from the larger and then find the difference between that and the smaller. Or, add 1 to the smaller and then subtract from the larger. Example: Between 7 and 15.
Method 1 15 - 7 = 8 8 - 1 = 7
Method 2 15 - 1 = 14 14 - 7 = 7
Method 3 7 + 1 = 8 15 - 8 = 7

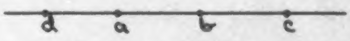
## Answers to Exercises 3-7

1. (a) 17 (b) 21 (c) 4 (d) 7 (e) None (f) 2 (g) None (h) 88 (i) 3 is answer. Discuss with class that the answer is either $a - (b + 1)$ or $(a - b) - 1$ or $(a - 1) - b$.
2. (a) 10 (b) 11 (c) 24 (d) 30 (e) 18 (f) 22 (g) 16 (h) 9
3. (a), (b), (c), (g), (i), (j).
4. (a) Yes. (b) Yes. (c) Yes.



5.





Either of the two situations is possible. The diagrams indicate that b is between c and d regardless of whether $c < d$ or $d < c$.

### 3-8. The Number One

In this lesson emphasis should be placed not only on special properties of the operations with 1 but also on the closure, associative and commutative properties. The fact that there is more than one way to represent the number 1 is emphasized in the first exercise. Of course this gives the teacher the opportunity to review the concept of numeral as a name for a number and not the number itself. Pupils think of the operation with numbers so frequently that they forget that (4-3) is really another way to represent 1.

A class discussion of the lesson summary in symbols may be profitable for many pupils. Of course some other letter in place of C could be used as practice for pupils in translating symbols into words. Be sure that the pupils have the ideas before attempting symbolism. Pupils' translations of the mathematical sentences could be somewhat as follows:

(a) The counting numbers (C) are obtained by starting with 1 and repeatedly adding 1.

(b) If any counting number is multiplied by 1, the product is the same number.

(c) If any counting number is divided by 1, the quotient is the same counting number.

(d) If any counting number is divided by the same number, the quotient is 1.

(e) The number one, raised to any power which is a counting number, equals 1.

## Answers to Exercises 3-8

1. The symbols in the following parts represent the number 1: (a), (b), (c), (d), (e), (i), (k), (l), (m), (o), (p).

2. (a) $100\cdot 1 = 100$

   (b) $10\cdot 1\cdot 1\cdot 1 = 10$

   (c) $\frac{14}{1} = 14$

   (d) $1\cdot\frac{2}{3}\cdot 1 = \frac{2}{3}$

   (e) $0\cdot 1 = 0$

   (f) $1\cdot 0 = 0$

3. We can get any counting number by the repeated addition of 1 to another counting number if the number we wish to get is larger than the counting number to which we add.

   We can get any counting number by the repeated subtraction of 1 from another counting number if the number we wish to get is smaller than the counting number from which we subtract.

4. Yes. $1 - 1 = 0$; $3 - 1 - 1 - 1 = 0$. Zero is not a counting number.

5. The successive addition of 1's to any counting number will give a counting number. But, the successive subtraction of 1's from any counting number will become 0 if carried far enough.

6. (a) 876429 (b) 976538 (c) 897638 (d) 896758
(e) 3479 (f) 97 (g) 1 (h) 1

## 3-9. The Number Zero

The purpose of this lesson is to understand why we can or cannot perform the fundamental operations with zero.

It is important for pupils to understand that zero is a perfectly good number and that it does not mean "nothing". The pupil should see that in addition and subtraction zero obeys the same laws as the counting numbers.

In explaining the product of $C \cdot 0$ and $0 \cdot C$, it may be helpful to review briefly the meaning of multiplication. Such discussion questions might be:

1. What is another way to find the answer to $3 \times 5$?
2. What does $3 \times 5$ mean? It means $5+5+5$ and not $3+3+3+3+3$.
3. Make up a real problem using $3 \times 5$. (The price of 3 pencils at 5¢ each.)
4. What does $5 \times 3$ mean? (It means $3+3+3+3+3$.)
5. Make up a real problem using $5 \times 3$. (The price of 5 pencils at 3¢ each.)

After such questions, zero may be introduced in the discussion as multiplicand and multiplier, since $5 \cdot 0 = 0 \cdot 5$ by the commutative property for multiplication.

In case of division, pupils should understand why we divide 0 by a and do not divide a by 0. It may be desirable to use several examples so that the pupils will see that 0/c should be 0 and c/0 is not the name of any whole number.

Some of the pupils may be interested in why we do not define c/0 = 1 or some other number. They should understand that it would be out of harmony with the fact that zero times any number equals zero.

The translations into words of the symbolic statements concerning zero can be somewhat as follows:

(a) If zero is added to any whole number, the sum is the same whole number.

(b) If any whole number is added to zero, the sum is the same whole number.

(c) If zero is subtracted from any whole number, the difference is the same whole number.

(d) If the product of any two whole numbers is zero, then one of the whole numbers is zero or both are zero.

(e) If any whole number is multiplied by zero, the product is zero.

(f) If zero is multiplied by any whole number, the product is zero.

(g) If zero is divided by any counting number, the quotient is zero.

(h) Zero cannot be used as a divisor.

## Answers to Exercises 3-9

1. The symbols in the following parts represent zero: (b), (d), (f), (h), (i), (j), (k), (l), (m), (n), (o), (q), (r), (s).

2. (a) 18424 (b) 641388 (c) 144, remainder 56
(d) 151, remainder 42 (e) $36538.26 (f) $60477.81
(g) 0 (h) $846.25 (i) $70.65 (j) 679 (k) 379
(l) 897 (m) $397.16 (n) Division by zero not possible
(o) 1 (p) $1846 (q) 0 (r) 0 (s) 0 (t) 0
(u) 976 (v) $97.46

3. The error is in the generalization to C in part (i). If $a \cdot b = C$, $a$ or $b$ does not need to be C. Example: $2 \cdot 2 = 4$. This exercise shows the error that may be made by making a generalization on a few cases.

## 3-10. Summary

(In student text only)

Answers to "How Are You Doing?" Questions, Chapters 1-3

1. $(122)_{three} = (17)_{ten} = (32)_{five}$ (It is easiest to get to base five by going through base ten.)

2. Yes. Start by filling either the 3-cup or the 5-cup container. If the three-cup is filled first then:
(a) Pour 3 into 5; (b) Fill 3; (c) Pour 2 from 3 into 5, which leaves 1 in 3; (d) Empty 5, pour 1 left in 3 into 5; (e) Fill 3. Now we have 4.
If the five-cup is filled first then:
(a) Pour 3 from 5 into 3, which leaves 2 in 5; (b) Empty 3 and pour in 2 from 5; (c) Fill 5; (d) Fill 3 by pouring 1 from 5; (e) Empty 3. Now we have 4.

3. $\frac{1}{6}$

4. 1,111
5. 176×176×176×176×176×176×176
6. If the sum of its digits is a multiple of 3.
7. Base 2
8. (2×27) + ([illegible]×9) + (1×3) + (0×1)
9. 31
10. 4×216 is the value of the 4 in $(4512)_{six}$ which equals 864.
    4×27 is the value of the 4 in $(41)_{twenty\text{-}seven}$, which equals 108.
    864 is 8 times 108.
11. The set of whole numbers has a zero and the set of counting numbers has not.
12. The statement is false. If one can find one example where the operation is not closed, then this operation is not closed for the set of whole numbers.
13. Commutative property of multiplication.
14. One (1)
15. Any of the properties listed in the section on zero.
16. 13·(2 + 5) or (2 + 5)·13
17. 136 + (25 + 75)
18. Multiply 65 times 11 and see if the product is 715.
19. 40
20. (7·3) + 6·(5·3)

## 3-11. Sample Test Questions for Chapter 3

1. Insert a symbol which makes a true statement:

   8 + 4     4 + 8

2. How many days are there between March 13, 1951 and March 27, 1951?

3. Show with one example that the set of numbers from 10 to 15 is not closed under addition.

4. Answer true or false: The identity for multiplication in the set of whole numbers is zero.

5. If K is a counting number then $\frac{0}{K}$ = ?

6. Apply the commutatative property of addition to: $(4 \cdot 5) + 6$

7. We are using the __________ property when we say that $3a + 5a$ is another way of writing $(3 + 5) \cdot a$.

8. If the product of 5 and a certain number is zero, then that number must be: (a) 1 (b) 0 (c) 5 (d) None of the above

9. When the number one is divided by any counting number n , the answer is always: (a) 0 (b) 1 (c) n (d) None of the above

10. Which of the following numerals are names of counting numbers?
$(10)_{two}$ 14 0 $(713)_{ten}$ $\frac{2}{3}$ XIV

11. $[(7 + 3) + 7 \cdot (6 \cdot 5)] = [(7 + 3) + (7 \cdot 6) \cdot 5]$ illustrates the associative property of ___?___

12. How many counting numbers are there between $(10)_{five}$ and $(13)_{four}$?

13. Make a true statement of $(3 \cdot 7) + (4 \quad) \quad 7 \cdot ( \quad )$

14. To check the statement $7 \times 345 = 2415$ by the inverse operation we would ___?___.

15. The letters a, b, and x represent counting numbers, and $\frac{a}{b} = x$. What can we say about the relation between a and b?

### 3-12. Answers to Sample Test Questions

1. =
2. 13
3. $12 + 13 = 25$
4. False
5. 0
6. $6 + (4 \cdot 5)$

7. Distributive

8. b

9. d

10. $(10)_{two}$, 14, $(713)_{ten}$, XIV

11. Multiplication

12. 1

13. $(3 \cdot 7) + (4 \cdot 7) = 7 \cdot (3 + 4)$

14. Divide 2415 by 7 (or by 345)

15. a is a multiple of b, and a is either greater than b or equal to b.

## CHAPTER 4

## NON-METRIC GEOMETRY

The principal objectives of this chapter are threefold:

(1) To introduce pupils to geometric ideas and ways of thought

(2) To give pupils some familiarity with the terminology and notation of "sets" and geometry, and

(3) To encourage precision of language and thought.

There is an attempt to guide the student to the discovery of unifying concepts as a basis for learning some of the more specific details. This chapter forms a background for Chapters 7, 9, and 10 which deal with metric or distance properties. It attempts to focus attention on ideas which are fundamental but which (while sometimes vaguely taken for granted) are often poorly understood by students.

Traditionally, these ideas have been taught when they were needed for a particular geometric discussion. But, all too often the teacher has assumed that these properties are obvious or clear without mentioning them. Also, there should be some advantage in considering together this group of closely related analagous properties and observing relations among them. The higher level study of some aspects of non-metric geometry has become a separate mathematical discipline known as projective geometry.

Reading the text. This chapter has been written with the in-tent that it be carefully read by students. We suggest that not only should students be assigned to read the material, but that they also be encouraged to study it. Reading mathematics is not like reading a novel. Students may find it necessary to "get in and dig" for ideas. After Section 4-2, students should be advised to read with a paper and pencil at hand so that they may draw diagrams to assist their understanding.

Precision of language. Pupils should be encouraged to express themselves accurately. Some pupils will be able to do much better than others.

Spatial Perception. It has been our effort here to help boys and girls develop spatial understanding. We do this in part by representations in space. We hope that suggestions given to the students and the notes here may be helpful in selecting other representations appropriate for your class. This course is not a 12th grade course. It is intended to provide background which is sometimes assumed in later courses.

Testing. In testing, try to test for grasp of ideas not for mere recall. Students should be encouraged to express ideas in their own words. Because abilities in geometric perception and understanding differ from abilities in arithmetic, you may observe some redistribution of high and low grades among your students.

Time Schedule. The following time schedule is suggested:

| | |
|---|---|
| 1st week | Sections 1 through 4 |

2nd week Sections 5 through 7 or 8

3rd week Remainder of chapter including test evaluation.

High ability classes may do the chapter in 11 days including the test.

Each section was conceived as a one-day section. It seems possible that sections 3 and 5 may take 2 days each and that an extra day or so might be spread over Sections 8, 9, 10. We believe all sections are important. We have tried to order the sections so that the ideas do not become more difficult. In fact, the second half of the chapter which deals less with spatial perception may well be easier for some students. Some students will particularly enjoy Sections 8 and 10. If a class is under great time pressure, Section 9 might be omitted. If so, it should be assigned to all the better students.

### Materials

Insights into ideas developed in this chapter will be greatly enhanced by use of instructional devices. Encourage students to make *simple* models as a means of helping in basic understandings. Emphasize ideas, not evaluation of models. As in using any instructional material of this kind, seek understanding of ideas without over-dependence upon representations.

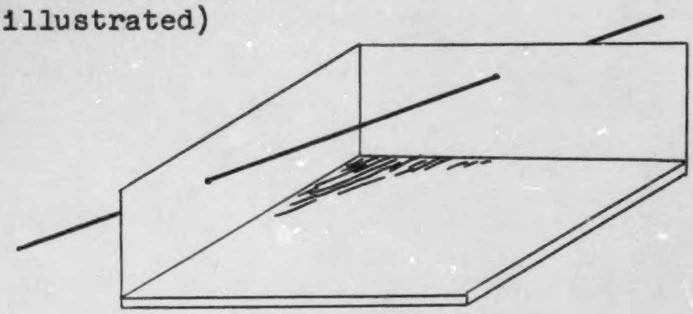
Suggested Materials:

String--to represent lines in space

Paper--to represent planes, and folded to represent lines and intersections of planes.

Tape or tacks for attaching string to walls, floor, other points in the room

Model (as illustrated)



Suggest making the model as shown above by using a cardboard carton (or, it can be made using heavy paper, oak tag, screen wire). Cut away two sides so that only 2 adjacent sides and bottom of box remain. String, wire, etc., may be used to extend through and beyond "sides", "floor", etc.

Oak tag for making models to be used by both teacher and students.

Coat-hanger wire, knitting needles, pickup sticks

Scissors, colored chalk

Light-weight paper (for tracing in exercises)

Yardstick or meter stick with several lengths of string tied to it at different intervals. By fastening stick to wall, lines may be represented by holding the string taut. By gathering together the free ends at one point the plane containing the point and the yardstick may be shown.

Optional:

Long pointer for seeing lines

Tinker-Toys for building models

Toothpicks for student models

Saran wrap, cellophane, and wire frame for representing planes.

## 4-1. Points, lines, and Space

1. Understandings:

(a) A point has no size.

(b) A line is a certain set of points.

(c) A line is unlimited in extent.

(d) Through two points there is one and only one line.

(e) Space is a set of points.

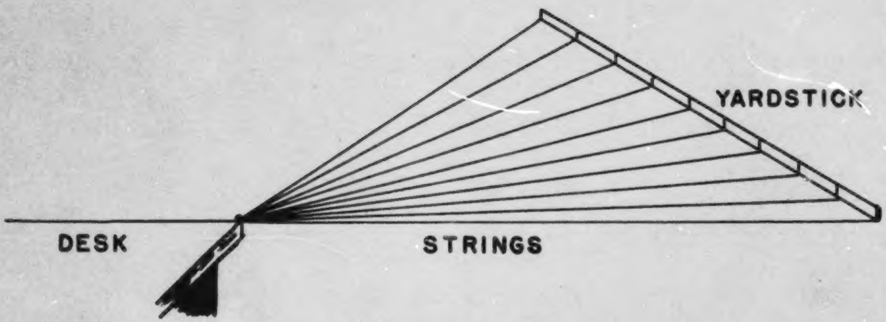
## 2. Teaching Suggestions

Just as we use representations to develop the concept of the "counting numbers" (2 cars, 2 people, 2 hands, 2 balls, 2 chairs, etc. to develop the concept of twoness) similarly we must select representations for developing the concepts of point, line, plane, and space.

Point. Identify things which suggest the idea of a point keeping in mind that one suggestion by itself is not adequate for developing the idea of a point. One needs to use many illustrations in different situations. Suggestions: tip of pencil, needle, needle, pointer; collection of boxes, putting one inside another and always being able to place one more inside of the last and the point indicated as being in all the boxes; pupil of the eye in intense brightness; progressive closure of shutter of camera; dot of light on some TV screens; particle of dust in the air.

Line. Identify two points using some of the situations as above, such as tips of two pencils, etc. Bring out the idea that given these two points there are many other points of the line that contain them. Some of these are between the two points, some are "beyond" the one, and some are "beyond" the other. Also, through two points there can be only one line. The line has no thickness and no width. It is considered to extend indefinitely. Use string held taut between two points to show representations of lines in positions that are horizontal, vertical, and slanting. Each student may represent lines by using a pencil between his fingertips. With each example talk about thinking of a line as unlimited in extent. Emphasize frequently that we use the word, "line," to mean straight line. Identify other representations of lines such as: edge of tablet (holding tablet in various positions); edge of desk; vapor trails; edges of roof of building; etc. It is important to select illustrations representing lines in space as well as the usual representations made by drawing on chalkboard and paper.

Space. Models will be most helpful here. Using "string on yardstick" and considering some point on a table, desk, or on some object which all students can see, let all the representations of lines from the yardstick pass through the point. Also, use string to show representations of other lines from other points on different walls, the floor, etc., all passing through the point. Use the model as described in drawing under "Suggested Materials". Pass lines (string, wire, thread) through "walls" and "floor" to suggest infinite number of lines and that these lines extend indef-



initely. Bring out the idea that each line is a set of points, and that space is made up of all the points on all such lines.

### Answers to Class Discussion Problems in 4-1

1. Depends on what objects are in the room.
2. The body of the porcupine is like the point. The quills are like lines through this point.

### Answers to Exercises 4-1

1. No, the ribbons are usually not straight.
2. There is exactly one line containing the two points on the monuments.
3. The line of the new string passes through the same two points as the line of the old. (Property 1)

## 4-2. Planes

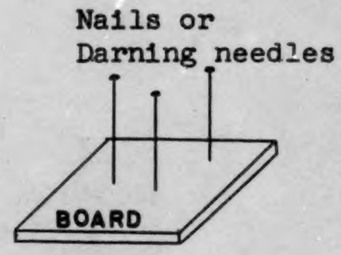
1. Understandings
   - (a) A plane is a set of points in space.
   - (b) If a line contains two different points of a plane, it lies in the plane.
   - (c) Many different planes contain a particular pair of points.

(d) Three points not exactly in a straight line determine a unique plane.

## 2. Teaching Suggestions

Identify surfaces in the room which suggest a plane--walls, tops of desks, windows, floor, sheet of paper, piece of cardboard, chalkboard, shadow. Make use of Saran wrap, cellophane, and a wire frame to show further a representation of a plane since this more nearly approaches the mathematician's idea of a plane. With each example bring out the idea that a plane has no boundaries, that it is flat, and extends indefinitely. It is an "ideal" of a situation just as is a line and a point. We try to give this idea by suggesting things that represent a plane. It is important to suggest representations of planes in horizontal, vertical, and slanting positions. Note that if a line contains 2 points of a plane, it lies in the plane and that many planes may be on a particular pair of points as pages of a book, revolving door, etc.

Then using three fingers or sticks of different heights in sets of 3 (not in a straight line) as suggested by the sketch at the right, see what happens when a piece of cardboard is placed on them. Add a fourth finger or a fourth stick and observe what happens.



Each student may try this experiment by using three fingers of one hand (and also three fingers using both hands) letting a plane be represented by a book, piece of oak tag, card. Change position of fingers and thumb by bending the wrist (changing sticks in model). Ask class to make a statement about three points not in a straight line (Property 3).

Demonstrate with wires or string the ideas in the last paragraph before asking students to read it or suggest that one or two students be responsible for demonstrating the idea to other members of the class.

The Class Discussion Problems may well be developed as a class activity.

A note. What is a basic motivation for the study of geometry? In our daily living we are forced to deal with many flat surfaces and with things like flat surfaces. It would be foolish not to note similarities of these objects. So, we try to note them. In so doing we try to abstract the notion of flat surface. We try to find properties that all flat surfaces have. Thus, we are led to an abstraction of the flat surface--the geometric plane. We study two aspects of this--(1) What a plane is like, considered by itself (plane geometry), and (2) how various planes (flat surfaces) can be related in space (one aspect of spatial geometry).

Just how do we study the geometric plane? We study it by thinking of what the plane is supposed to represent, namely, a flat surface. However, in trying to understand a plane (or planes), we find it difficult to think just abstractly. Thus, we think of representations of the plane--wall, chalkboard, paper, etc., and we think of these as representations of the abstract idea. The abstract idea enables us to identify characteristics which all flat surfaces have in common.

## Answers to Questions in Text, 4-2

Yes, this is what Property I says. Yes, the lines must be on the plywood. Hold palm flat and move arm and hand in plane desired. Yes, new position, and no, only one way. Three points are in one plane; the fourth may not be in it. Property 2.

## Answers to Class Discussion Problems

1. Depends on the particular classroom.
2. There are many different sloping positions, illustrating Property 2.

   Only one, illustrating Property 3.

## Answers to Exercises 4-2

1. All in the same line
2. Not all in the same line
3. All legs must rest on the same plane or the device may rock. Three points not all on the same line are in

exactly one plane.

4. (a) Many (unlimited quantity) (b) One

5. (a) Many (b) Many (c) Exactly one, unless the points are all on the same line.

6. Wing surface, flat surface used on water. Consult dictionary.

*7. (a) Six, as long as no three are on the same line

(b) Six

8. Explanation in Section 4-5

## 4-3. Notation and Symbolism

1. Understandings

(a) Students learn to recognize how planes, lines on planes, lines through planes, etc. are represented by drawings.

(b) Students learn to name particular points, lines, and planes using letters, etc.

(c) Students learn how to interpret and understand perspective drawings.

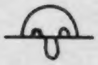
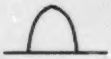
(d) Students learn to develop an awareness of planes and lines suggested by familiar objects.

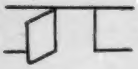
2. Teaching Suggestions

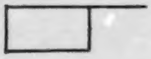
Bring out the idea that we make agreements as to how to represent certain ideas, i. e. "." for a point, " ╱ or ⤢ for a line, and the use of letters for naming lines and points.

Note that we usually name points by capital letters, lines by lower case letters or pairs of capital letters with bar and arrows above, as, $\overleftrightarrow{AB}$, and that a plane is named by three capital letters or a single capital letter. Also, we sometimes talk about two or more lines, planes, etc. by using subscripts, such as, $\ell_1$, $\ell_2$, and $\ell_3$.

We do not expect students to learn to make drawings showing more than one plane, intersections of planes, etc., but we want them to learn to interpret drawings. However, if some students wish to attempt such drawings, certainly do not discourage them.

Students enjoy a guessing game about abstract figures such as these:  "man looking over fence" or  "boat sinking." They might enjoy a similar game with planar abstractions such as:

 "open door" or "pup tent" or

 "butcher knife".

## Answers to Exercises 4.3

1. A is to the left , B to the right

   C is nearest the top , D nearest the bottom

2. A table

3. It has turned upside down

4. (a) Cot (g) Line of laundry

   (b) Ping pong table (h) Open door

(c) Football field

(d) Carpet

(e) High jump

(f) Coffee table

(i) Chair

(j) Shelf

(k) Ladder

5. Yes (or no), We assume point V is on line $\overleftrightarrow{PQ}$ and not "behind" $\overleftrightarrow{PQ}$

No

No

One, P.

6. 1, b

2, d

3, c

4, d

5, a

6, a

7. Yes, Yes

Yes, $\ell$ is the only line through P and Q

$\ell_2$ is the intersection of $M_1$ and $M_2$

No, No, No,

## 4-4. Intersection of Sets

1. Understandings

(a) A set contains elements which are collected according to some common property or explicit enumeration.

(b) The common elements in two or more sets make up the elements of the intersection of two or more sets.

2. Teaching Suggestions

Review the idea of sets by asking students to describe certain sets, as set of names of members of the class, the set of members of the class, set of all students in class whose last name begins with "B", set of even numbers, set of counting numbers between 12 and 70 having a factor 7 (i.e. 14, 21, 28, ... ).

Explain that any 2 sets determine a set which is called their intersection, that is, the set of elements (if any) which are in both sets. Have students give the intersection for the set of odd numbers between 1 and 30 and the set of counting numbers having the factor 3 between 1 and 30. Note the three sets--the two given sets and the intersection of the two sets. Use other illustrations such as set of boys in the class and the set of students with brown eyes. Also, find intersection of three sets--students in class having blonde hair, students in class having birthdays in November, and students in class riding the school bus. In selecting sets, include some in geometry (i.e. the intersection of two lines in the same plane, etc.).

Note that the empty set is the intersection of two sets with no elements in common.

After developing the idea of intersection go back to examples and describe how the idea can be expressed in symbols. It is a code we can use and like many codes it simplifies the expression. For example,

Set A = {3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29}

Set B = {3, 6, 9, 12, 15, 18, 21, 24, 27}

$A \cap B$ = {3, 9, 15, 21, 27}

You may wish to refer to the supplementary unit on "sets". It will provide further symbolism for better students and/or those who especially enjoy using symbols.

## Answers to Exercises 4-4

1. (a) {18, 19, 20, 21, 22}
   (b) As in your state
   (c) The empty set
2. (a) 1, 3, 5 (a possible answer)
   (b) 5, 10, 15 (a possible answer)
   (c) P, R, T (a possible answer)
3. (a) 9, 10, 11, 12
   (b) As in your class
   (c) The point P
   (d) The line
4. S T = {10, 15, 23}
5. (a) An edge
   (b) The empty set
6. (a) S = {Maine, N. H., Vt., Conn., Mass., R. I.}

T = {New Hampshire, New York, New Jersey, New Mexico}

V = {Texas, New Mexico, Arizona, California}

(b) New Hampshire

(c) The empty set

(d) New Mexico

7. (a) Yes

(b) Yes. The addition of multiples of 5 gives multiples of 5.

8. Each part is almost obvious from the notions of set and intersection.

(a) This is by definition since the empty set is a set.

(b) The set of those elements in X which are also in Y is the set of those elements in Y which are also in X.

(c) Each "side" of the equality means the same set: namely the set of all elements which are in each of the three sets X, Y, and Z.

## 4-5. Intersections of Lines and Planes

1. Understandings

(a) Two lines may:

(1) be in the same plane and intersect.

(2) be in the same plane and not intersect (intersect in the empty set).

(3) not be in the same plane and not intersect (intersect in the empty set).

(b) A line and a plane may:

(1) not intersect (intersect in the empty set).

(2) intersect in one point.

(3) intersect in a line.

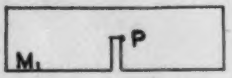
(c) Two different planes may

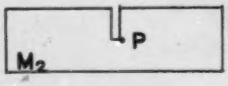
(1) intersect and their intersection will be a line.

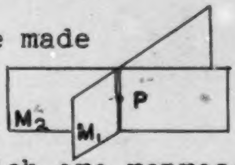
(2) not intersect (have an empty intersection).

2. Teaching Suggestions

Use models in order to explore the possible situations for two lines intersecting and not intersecting. (Let each student have materials, too.) Also, use a pencil or some other object to represent a line, and a card to represent a plane. Use two pieces of cardboard each cut to center with the two fitted together to represent the idea of two planes and their intersection, and, from these, state some generalizations that may be made







Also, identify situations in the room which are representations of different cases of intersections of lines and planes. Some may wish to express the ideas in symbols. For example, name the line of one of the front vertical edges   , and line along the top front of desk, $k$. Then $l \cap k$ is a point.

Subscripts also may be used so that we talk about lines $l_1$ and $l_2$. The use of a few subscripts should be encouraged.

## Answers to Questions in Text 4-5

Indefinite question - no specific answer needed. Yes it is true of edges of classroom. Yes, yes, yes, no. Yes, Figure from Section 4-3 could be used.

Case 1. If they did, $\ell$ and k would be the same line.

Yes, (b) line and plane with one point intersection

(a) line lying in plane

No; no. Cannot hold sheets to do either.

No other case; either the planes intersect or they don't.

## Exercises 4-5

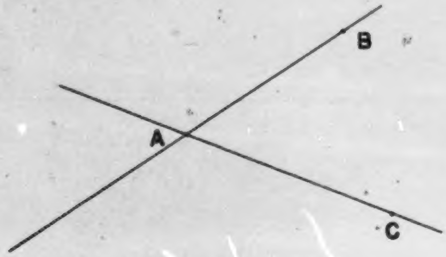
1. (a) Intersection of line and plane, empty set. Line and plane are parallel. No point.

(b) Intersection of line and plane, one point line pierces plane.

(c) Intersection of line and plane more than one point. Line lies in plane.

2. Intersection of floor and front wall, intersection of ceiling and side wall, etc.

3.



$AB \cap AC$ = Point A

4. Yes, yes, no (no point in all planes), the empty set, no.

5. Yes, yes, one point

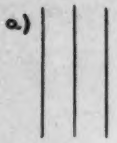
6. Yes, yes, a line (the fold)

7. (4) The empty set

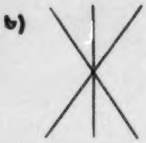
   (5) One point

   (6) The line

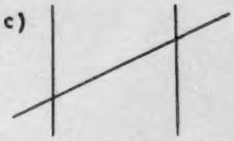
8.



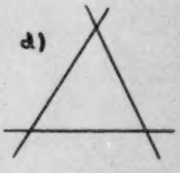
0 points



1 point



2 points



3 points

Using Only Lettered Points

9. (a) HGD and ABC (one other possibility)

   (b) HGB and GBC (many other possibilities)

   (c) HGB, BGD, FGD (many other possibilities)

   (d) HGD, FGD, BGD (some other possibilities)

   (e) ABC, FE (many other possibilities)

   (f) FE, GD (many other possibilities)

   (g) ED, AB (many other possibilities)

   (h) ED, GD, CD (many other possibilities)

   (i) FGB, FGD, HGD, BGD

## 4-6. Segments

1. Understandings

(a) If A, B, and C are three points on a line, our intuition tells us which point is between the other two.

(b) A line segment is determined by any two points and is on the line containing those points.

(c) The two points which determine a segment are called endpoints of the segment.

(d) A line segment is a set of points which consists of its endpoints and all points between them.

(e) The union of two sets consists of all the elements of the two sets.

## 2. Teaching Suggestions

Bring out the idea that when we draw a sketch or a picture of a line, we draw a picture of one part of the line, and that this is, properly, a line segment. However, we often represent a line by a part of a line (since we cannot do anything else). One should be careful to say that the sketch represents a line or line segment as is appropriate. Draw a representation of a line on the chalkboard and name two points of the line, A and B. Note that $\overline{AB}$ means points A and B and all points between them. Name other points on the line and various line segments.

Review the idea of intersection of two sets. Then develop the idea of union of two sets using line segments much as is suggested by student text. Extend this idea by selecting other illustrations, as, set of girls in class and set of boys in class and that the union of the two sets is the set obtained by combining the two sets.

Each member of the union is then either a boy or a girl in the class. Have students suggest other sets, noting what can be said about the elements of the union of the two or more sets. Take illustrations from geometry too, such as, the union of two lines, line and plane, etc.

Some students may ask about expressing these ideas in symbols. If so, you may wish to again refer to the supplementary unit on "sets". Also, express one of illustrations in symbols, as,

Set A = {4, 8, 12, 16, 20, 24, 28}

Set B = {6, 12, 18, 24}

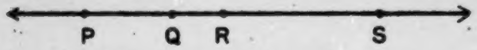
A  B = {4, 6, 8, 12, 18, 20, 24, 28}

Again note that if an element is a member of both sets, it appears only once in the their union.

## Answers to Exercises 4-6

In text $\overline{BC}$

1.



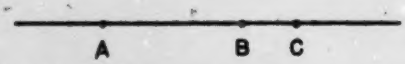
(a) $\overline{PR}$, $\overline{QS}$ or $\overline{PR}$, $\overline{QR}$ etc.

(b) $\overline{PQ}$, $\overline{QR}$ etc.

(c) $\overline{PQ}$, $\overline{RS}$

(d) $\overline{PQ}$, $\overline{RS}$

2.



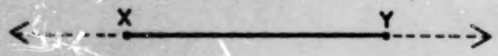
(a) The point B

(b) $\overline{BC}$

(c) $\overline{BC}$

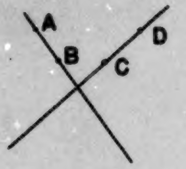
(d) $\overline{AC}$

3. No;

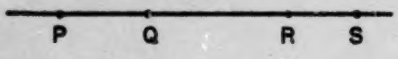


yes

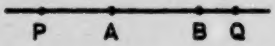
4.



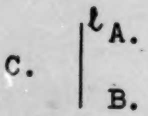
5.



6. No.



7.



Yes; no.

8. We would have more than one "straight line" through two points. See question 6.

## Answers to Questions in Text 4-7

Beginning below the first figure of the section:

Yes, no, not the empty set. A half-line is one of the two sets of points into which a line is separated by a point. Yes, a point; yes (a set consisting of one point). No; yes; yes for figure. See statement (3) immediately below in text.

## 4-7. Separations

1. Understandings

   (a) A plane separates space into two half-spaces.

   (b) A line separates a plane into two half-planes.

   (c) A point separates a line into two half-lines.

   (d) A ray is the union of a half-line and the point which determines the half-line.

2. Teaching Suggestions

Use cardboard models to develop understanding of these ideas. Note how one side of the model pictured in the materials section of this Commentary separates space and that we call the two sides half-spaces. Ask how we could separate a plane into two half-planes and a line into two half-lines before the students have read these paragraphs.

This section gives an unusually good opportunity to emphasize relations among point, line, plane, and space. You can expect seventh grade students particularly to enjoy this section. It gives a certain structure to geometry.

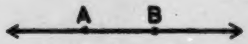
Draw a number of lines on the chalkboard. Mark points on them and talk about half-lines, rays, and endpoints. Talk about the intersection of 2 rays, two half-lines, and ray and half-line. If students are worried as to whether a half-line has an endpoint, see explanation in student text for saying "an angle of a triangle."

Also, identify representations of half-spaces, produced by room dividers, walls in building; of half-planes, by lines on paper, lines on wall, etc.; and of half-line by naming a particular point along the edge of a ruler.

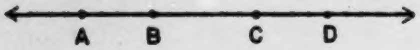
## Answers to Exercises 4-7

1. Yes; no; no; no; yes; yes.

2. $\overline{AB}$ (or $\overline{BA}$); a half line to the left of A.



3.



(a) $\overrightarrow{BA}$, $\overrightarrow{BC}$

(b) $\overrightarrow{BA}$, $\overrightarrow{CD}$

(c) $\overrightarrow{BC}$, $\overrightarrow{CD}$

(d) $\overrightarrow{BA}$, $\overrightarrow{BC}$

(e) $\overrightarrow{BA}$, $\overrightarrow{CD}$

4. No, no

5. Yes, yes

6. Yes

*7. Not if they are half-planes from the same line, but yes if you use parallel lines and overlapping half-planes

*8. No; yes

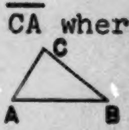
## 4-8. Angles and Triangles

1. Understandings

(a) An angle is a set of points consisting of 2 rays not both on the same straight line and having an endpoint in common.

(b) An angle separates the plane containing it.

(c) A triangle is the union of three sets, $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ where A, B, and C are any points not on the same line.



(d) A triangle determines its angles but does not contain its angles.

2. Teaching Suggestions

Illustrate the idea of angle as two rays with the same endpoint. Use colored chalk to show interior and exterior. Also note how we name an angle. (This is not the only way to consider an angle. This definition, however, is identical with that which is very likely to be used in the 10th grade. Later, the measure of an angle and angle as a rotation will be considered.

In developing the idea of triangle, put three points on board and note them as endpoints of 3 line segments, $\overline{AB}$, $\overline{BC}$, $\overline{AC}$. Note the set of points in each segment and that a triangle is the union of these three sets. Use colored chalk to show interior and exterior. Emphasize the set of points of the interior, the exterior, and that of the triangle.

Again students may be interested in drawing angles, triangles, shading, etc. (This is not perspective drawing.) This is good material to develop imagination and to encourage drawing etc. How many triangles can be drawn with just four lines? How many with four lines two of which are parallel? Etc.

In discussing the angles of the triangle bring out the idea that although people often talk about angles of a triangle, it is a short way of saying that they are the angles determined by the triangle. Use such analogies as the school determines its graduates but graduates are not in school; a city "has" suburbs, but the suburbs are not part of the city, etc.

### Answers to Questions in Text 4-8

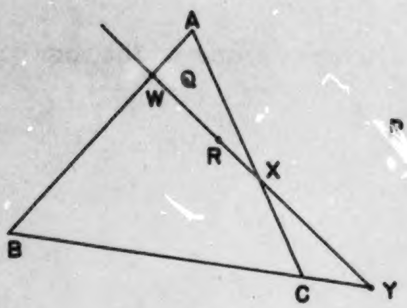
Yes, the sides of the triangle are in the triangle. That is to say, every point of each side is in the set of points which is the triangle. No, the angles are not in the triangle.

### Answers to Exercises 4-8

1. (a) The interior of $\angle ABC$
    (b) The interior of $\triangle ABC$
2. (a) Yes, they have different vertices
    (b) Yes
    (c) The lines containing the rays determine a plane.
3. (a) The point A
    (b) No
    (c) $\overline{AB}$
    (d) $\overline{AB}$
4. (a) The two points X and W
    (b) $\triangle AWX$, $\triangle ABC$, $\triangle BWY$, $\triangle XCY$
    (c) None
    (d) A, B, C, W and Y

(e) B , A

5. (a)

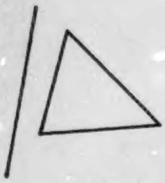


(b)

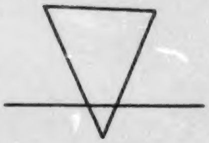
(c)

R can be on $\overline{WX}$

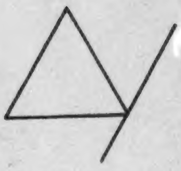
6. (a)



(b)

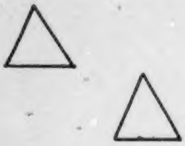


(c)

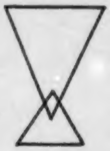


(d) Not possible

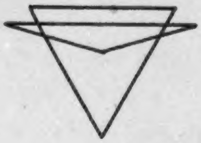
7. (a)



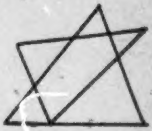
(b)



(c)



(d)



8. (a) The points A and C

(b) $\overline{AB}$

(c) The points A and B

(d) The point B

(e) $\angle ACB$

(f) $\overline{BC}$

(g) $\overline{BC}$

(h) The union of $\overline{AB}$ and $\overline{BC}$

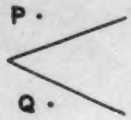
*9. No; yes.

*10. (a) Yes

(b) It may or may not, depending on choice of P and Q.

(c) Yes

(d) No



## 4-9. One-to-one Correspondence

1. Understandings

(a) The idea of one-to-one correspondence is fundamental in counting. (Chapter 3)

(b) One-to-one correspondences in geometry can be established:

(1) Between a certain set of lines and a certain set of points

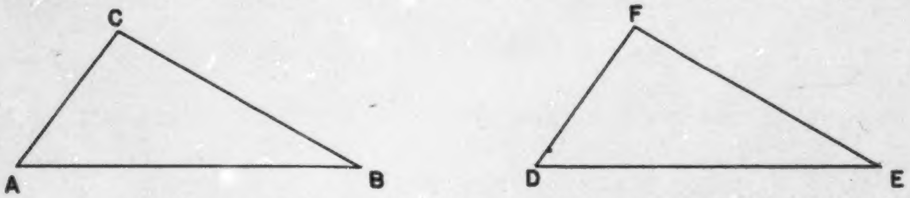
(2) Between the set of points of one segment and the set of points of another segment

(3) Between certain other geometric sets in pairs.

2. Teaching Suggestions

Review the idea of one-to-one correspondence and the necessary condition that for each element in set A there corresponds an element in set A. For example, if there are 5 chairs and 5 people, for each chair there is a person and for each person there is a chair.

This idea, while elementary, is sometimes hard to grasp. One-to-one correspondences between finite sets (sets having a specific number of elements as in the illustration above) are easy to observe if they exist. If sets A and B are finite, then if A and B have the same number of elements, there exists a one-to-one correspondence between them. Such follows simply by counting. But we are sometimes interested in a particular one of the possible one-to-one correspondences. For the two congruent triangles below



we are interested in matching A with D, B with E, and C with F. It is on such basis that we get the congruence. If we were to match A with F, B with D, and C with E we would not be noting the congruence.

For infinite sets H and K we may be interested in two aspects:

(1) Is there any one-to-one correspondence between H and K?

(2) Is there a "nice" or "natural" one-to-one correspondence?

In the examples in Section 4-9, we not only show that there is some one-to-one correspondence but that there is a "natural" or "nice" one. There also would be a great many that are not "natural" or "nice".

To establish a one-to-one correspondence we need (1) a complete matching scheme, and (2) in this particular device it must be true that for any element of either set there corresponds a unique element of the other set. It is implied by what we say that if a corresponds to *b*, then *b* corresponds to *a*.

In effect, to establish a one-to-one correspondence we must have a way of "tying" each element of either set to a particular element of the other. And the "string" we use for tying *a* to *b* also ties *b* to *a*.

In the text we describe a one-to-one correspondence between a certain set of lines and a certain set of points.

Problems 5 and 6 of the exercises are developmental and because of this are suggested for possible classwork.

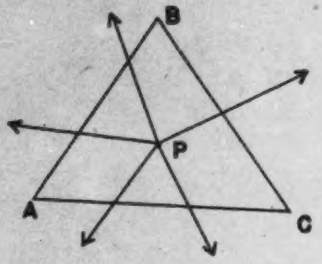
### Answers to Questions in Text 4-9

Yes (if all ticket-holders show up). Same number. Yes; yes; yes. One more person than chairs until winner is determined. Yes, at end of game. Yes.

### Exercises 4-9

1. That depends on the class and classroom.
2. No. There are more states than there are cities of over 1,000,000 in population.
3. Each vertex to the side opposite. Yes
4. $$\begin{array}{ccccc} 1 & 3 & 5 & 7 & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 0 & 2 & 4 & 6 & \cdots \end{array}$$ each odd to next lower even.

5.



Yes, you should observe it.

Yes; yes.

6. Yes; yes; yes; yes; yes

7. Yes, match elements of R with elements of T by means of element in S.

*8. Similar to rays in space. Otherwise as in Problem 5.

*9. The set of intersections of the elements of K with a plane form the set H.

*10. (a) Yes

(b) Yes

(c) Yes, see Problem 7.

11. Each whole number is matched with the whole number which is twice its value. Thus

$$\begin{array}{cccccc} 0 & 2 & 4 & 6 & 8 & \dots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 0 & 1 & 2 & 3 & 4 & \dots \end{array}$$

## 4-10. Simple Closed Curves

1. Understandings

(a) Broken-line figures such as those we see in statistical graphs, triangles, rectangles, as well as circles, and figure eights are curves.

(b) A simple closed curve in the plane separates the plane into 2 sets--the points in the interior of the curve and the points on the exterior of the curve. The curve itself is contained in neither set.

(c) The curve is called the boundary of the interior (or the exterior).

(d) If a point A is in the interior of a curve and a point B is in the exterior of the curve, then the intersection of $\overline{AB}$ and the curve contains at least one element.

2. Teaching Suggestions

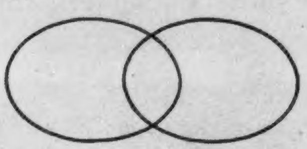
Draw some curves on the chalkboard, bringing out the idea that we call them "curves" and that a segment is just one kind of curve. We use the word "curve" in a special way in mathematics.

Note that a simple closed curve separates a plane into two sets and that the curve itself is the boundary of the two sets. Also, that any quadrilateral, parallelogram or rectangle is a simple closed curve. Identify some of the many curves which are suggested in the room, such as boundary of chalkboard, total boundary of floor surface, etc.

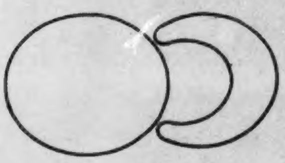
Students enjoy drawing elaborate curves which may still be classified as simple closed curves. Encourage their drawing a few simple closed curves for a bulletin board exhibit.

## Answers to Exercises 4-10

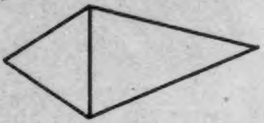
1.



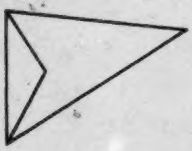
or maybe

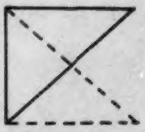


2. The intersection of the exterior of $J_1$ and the interior of $J_2$.

3. 

OR



[ 

This is ruled out because of the extra point of intersection. ]

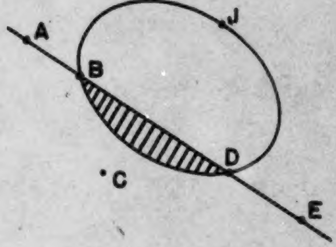
Yes, 3

4. No, it contains two.

5. Simple closed curve, line, angle

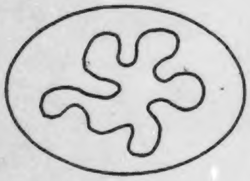
6. (a) B and D

(b)

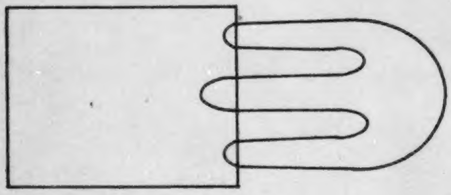


(c) $\overline{BA}$ and $\overline{DE}$

*7.



*8.

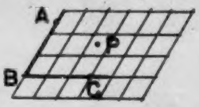


9. Because the curve would have to cross an even number of times. If the curve crosses three times the pencil is then not on the same side of the line as where it started.

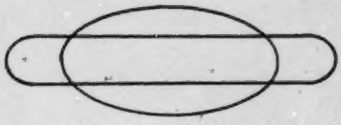
## Sample Test Questions on Chapter 4 (With Answers)

1. If two different planes each contain the same three points what can you say about the three points? (The points are on a line.)

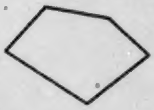
2. Draw ∠ABC. Label P a point in the interior. Shade the P-side of line $\overleftrightarrow{AB}$.



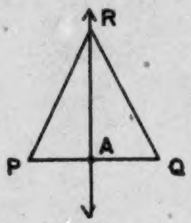
3. Draw two simple closed curves whose intersection is a set of exactly four points.



4. Draw a simple closed curve which is the union of five segments, no two on the same line.



5. Draw PQR. Label a point A between P and Q. Draw the line $\overline{AR}$. List all triangles represented in your figure. ( △PQR, △PAR, △RAQ.)

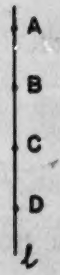


6. Each point of a line separates the line into two half-lines. Make two similar statements, one about a line and a plane and the other about a plane and space. (A line separates a plane into two half-planes. A plane separates space into two half-spaces.)

7. Suppose the intersection of two lines is the empty set. If they are in the same plane they are ( parallel ). If they are not in the same plane they are ( skew ).

lines $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{BC}$ separate the plane? (7)

12. Draw a vertical line $\ell$ on your paper. Label points A, B, C, and D in that order from top to bottom.



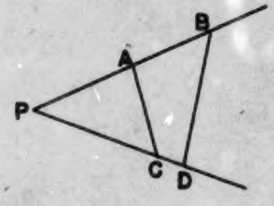
(a) What is $\overline{AC} \cap \overline{BD}$? ($\overline{BC}$)

(b) What is $\overrightarrow{BA} \cap \overrightarrow{CD}$? (The empty set)

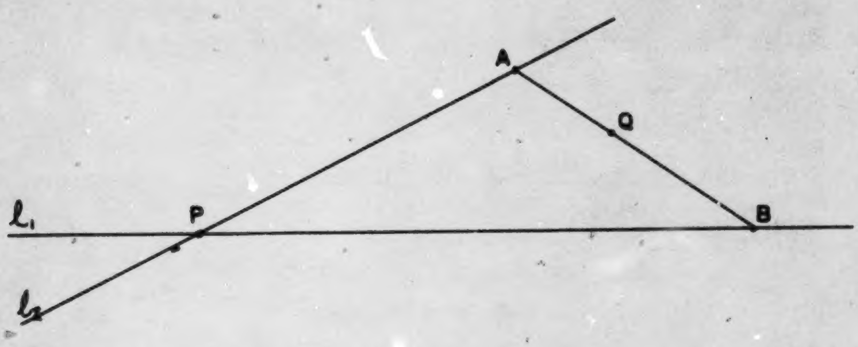
(c) What is the union of $\overline{AB}$ and $\overrightarrow{BC}$? ($\overrightarrow{AB}$)

(d) What is the union of $\overline{AD}$ and $\overline{BC}$? ($\overline{AD}$)

13. In the figure, a set of rays from P may be used to establish a (one-to-one correspondence) between $\overline{AC}$ and $\overline{BD}$.



14.



To the left of an item in the left-hand column place the letter of a corresponding item from the right-hand column:

| | | | | |
|---|---|---|---|---|
| (g) | 1. | the union of $\overrightarrow{PA}$ and $\overrightarrow{PB}$ | a. | $\overrightarrow{AP}$ |
| (f) | 2. | $\ell_1 \cap \ell_2$ | b. | $\overline{AB}$ |
| (c) | 3. | segment AQ | c. | $\overline{AQ} \cap \overline{AB}$ |
| (d) | 4. | point in interior of $\angle APB$ | d. | Q |
| (e) | 5. | $\overleftrightarrow{PB}$ | e. | $\ell_1$ |

8. Consider the figure at the right

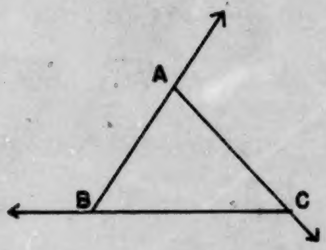
(a) List 3 rays represented

($\overrightarrow{BA}$, $\overrightarrow{AC}$, $\overrightarrow{CB}$)

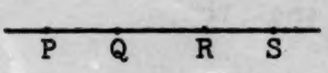
(b) What is $\overrightarrow{AC} \cap \triangle ABC$? ($\overline{AC}$)

(c) What is $\overline{AB} \cap \overrightarrow{AC}$?

(Point A)



9. Draw a horizontal line. Label four points on it P, Q, R, and S in that order from left to right.



(a) Name two segments whose intersection is one point.

($\overline{PQ}$ and $\overline{QR}$)

(b) Name two rays whose union is the line. ($\overrightarrow{PQ}$ and $\overrightarrow{QP}$)

(c) Name two segments whose intersection is a segment.

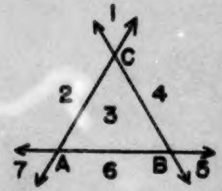
($\overline{PR}$ and $\overline{QS}$)

(d) Name two segments whose union is not a segment.

($\overline{PQ}$ and $\overline{RS}$)

10. Using the planes of the walls and ceiling of your room, list three planes whose intersection is exactly one point. (Foot, wall, side wall, and ceiling.)

11. Label A, B, and C three points not all on the same line. Draw $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{BC}$.



(a) Into how many pieces (regions) does the union of the segments $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ separate the plane? (2)

(b) Into how many pieces (regions) does the union of the

(a) 6. ray on $\ell_2$

(b) 7. the union of $\overline{AQ}$ and $\overline{QB}$

(i) 8. $\overline{QB} \cap \ell$

(h) 9. $\overline{AQ} \cap \ell_1$

(j) 10. the union of $\ell_2$ and $\overline{AP}$

f. P

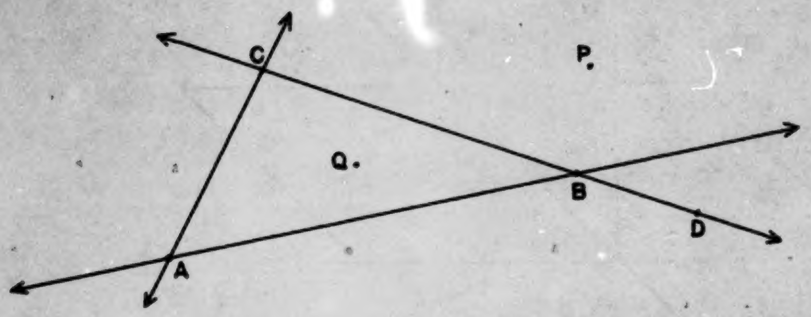
g. $\angle APB$

h. empty set

i. B

j. $\ell_2$

15.



In the corresponding blank to the left of each of the following statements indicate if it is true or false.

(True) 1. Point Q is on the C-side of $\overleftrightarrow{AB}$.

(False) 2. $\overleftrightarrow{AB}$ is the intersection of two half-planes.

(False) 3. The union of $\overrightarrow{CD}$ and $\overrightarrow{CA}$ is $\angle ABC$.

(False) 4. Point B separates $\overrightarrow{CD}$ into two segments $\overline{CB}$ and $\overline{BD}$.

(True) 5. Point B separates $\overline{CD}$ into two half-lines.

(True) 6. Point Q is in the interior of angle ACB

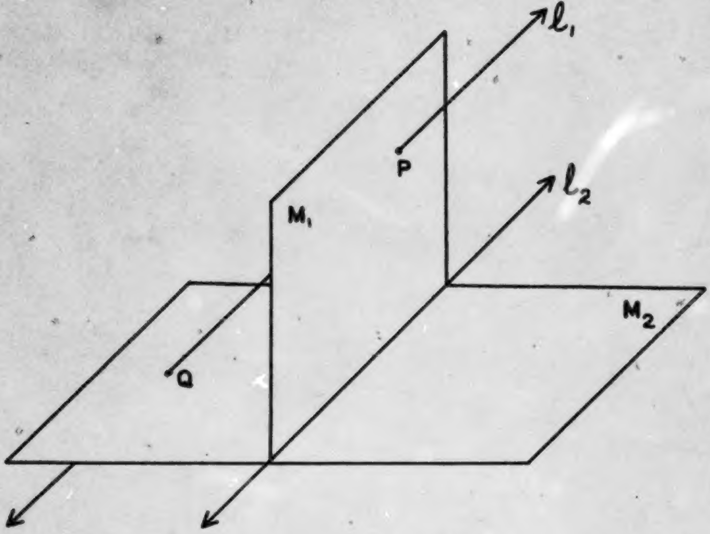
(False) 7. Point Q is in the interior of angle DBA

(True) 8. The interior of angle ABC is the intersection of a half-plane containing C and a half-plane containing A.

(True) 9. Triangle ABC is a simple closed curve.

(True) 10. A one-to-one correspondence may be established between (a) the set of lines intersecting $\overleftrightarrow{AC}$ and containing B and (b) the set of points on $\overleftrightarrow{AC}$.

16.



Multiple Choice.

1. $\ell_1$ and $\ell_2$:

(a) intersect in the empty set

(b) are parallel

(c) intersect in one point

(d) are in the same plane 1 (a)

2. Segment $\overline{QP}$:

(a) lies in plane $M_2$ (b) is in plane $M_2$

(c) connects an element of $M_1$ with an element of $M_2$ (d) none of these 2 (c)

3. (a) $\ell_2$ separates $M_1$ into two half-planes

(b) $\ell_2 \cap M_2$ is the empty set

(c) $\ell_2 \cap M_1$ is the empty set

(d) $\ell_2$ separates space 3 (a)

4. (a) $M_2 \cap \ell_1 = P$

(b) $M_2 \cap M_1 = \ell_2$

(c) $M_2$ does not extend endlessly as does $\ell_2$

(d) $M_2$ contains all the points of $\ell_1$ 4 (b)

5. Point P:

(a) is in the intersection of $M_1$ and $M_2$

(b) lies in a half-plane whose boundary is $\ell_2$

(c) is not an element of $\overline{PQ}$.

(d) is not an element of $\ell_1$ 5 (b)

# CHAPTER 5

# FACTORING AND PRIMES

## 5-1. Primes

In making the "sieve of Eratosthenes", the pupils have the opportunity to discover the set of prime numbers less than 100. At the same time they are learning to use the word "multiple". It will save class time for the teacher to have prepared on ditto sheets the numbers from 1 to 100 arranged in columns of 20. In view of the questions following Exercise 12, the numbers from 100 to 200 could be included on the sheet. It would be wise to separate the two sets of numbers. The word "sieve" has been used because all of the numbers that are not prime numbers, except the number one, are removed and prime numbers remain. It should be stressed that the number one is a special number that is neither prime nor a multiple of a prime number. The number two is the only even number that is a prime number. Multiples of 7 are the last ones that need to be crossed out.

## Sieve of Eratosthenes

Numbers 1 to 100

| | | | | |
|---|---|---|---|---|
| 1 | ~~21~~ | 41 | 61 | ~~81~~ |
| 2 | ~~22~~ | ~~42~~ | ~~62~~ | ~~82~~ |
| 3 | 23 | 43 | ~~63~~ | 83 |
| ~~4~~ | ~~24~~ | ~~44~~ | ~~64~~ | ~~84~~ |
| 5 | ~~25~~ | ~~45~~ | ~~65~~ | ~~85~~ |
| ~~6~~ | ~~26~~ | ~~46~~ | ~~66~~ | ~~86~~ |
| 7 | ~~27~~ | 47 | 67 | ~~87~~ |
| ~~8~~ | ~~28~~ | ~~48~~ | ~~68~~ | ~~88~~ |
| ~~9~~ | 29 | ~~49~~ | ~~69~~ | 89 |
| ~~10~~ | ~~30~~ | ~~50~~ | ~~70~~ | ~~90~~ |
| 11 | 31 | ~~51~~ | 71 | ~~91~~ |
| ~~12~~ | ~~32~~ | ~~52~~ | ~~72~~ | ~~92~~ |
| 13 | ~~33~~ | 53 | 73 | ~~93~~ |
| ~~14~~ | ~~34~~ | ~~54~~ | ~~74~~ | ~~94~~ |
| ~~15~~ | ~~35~~ | ~~55~~ | ~~75~~ | ~~95~~ |
| ~~16~~ | ~~36~~ | ~~56~~ | ~~76~~ | ~~96~~ |
| 17 | 37 | ~~57~~ | ~~77~~ | 97 |
| ~~18~~ | ~~38~~ | ~~58~~ | ~~78~~ | ~~98~~ |
| 19 | ~~39~~ | 59 | 79 | ~~99~~ |
| ~~20~~ | ~~40~~ | ~~60~~ | ~~80~~ | ~~100~~ |

Numbers 101 to 200

| | | | | |
|---|---|---|---|---|
| 101 | ~~121~~ | ~~141~~ | ~~161~~ | 181 |
| ~~102~~ | ~~122~~ | ~~142~~ | ~~162~~ | ~~182~~ |
| 103 | ~~123~~ | ~~143~~ | 163 | ~~183~~ |
| ~~104~~ | ~~124~~ | ~~144~~ | ~~164~~ | ~~184~~ |
| ~~105~~ | ~~125~~ | ~~145~~ | ~~165~~ | ~~185~~ |
| ~~106~~ | ~~126~~ | ~~146~~ | ~~166~~ | ~~186~~ |
| 107 | 127 | ~~147~~ | 167 | ~~187~~ |
| ~~108~~ | ~~128~~ | ~~148~~ | ~~168~~ | ~~188~~ |
| 109 | ~~129~~ | 149 | ~~169~~ | ~~189~~ |
| ~~110~~ | ~~130~~ | ~~150~~ | ~~170~~ | ~~190~~ |
| ~~111~~ | 131 | 151 | ~~171~~ | 191 |
| ~~112~~ | ~~132~~ | ~~152~~ | ~~172~~ | ~~192~~ |
| 113 | ~~133~~ | ~~153~~ | 173 | 193 |
| ~~114~~ | ~~134~~ | ~~154~~ | ~~174~~ | ~~194~~ |
| ~~115~~ | ~~135~~ | ~~155~~ | ~~175~~ | ~~195~~ |
| ~~116~~ | ~~136~~ | ~~156~~ | ~~176~~ | ~~196~~ |
| ~~117~~ | 137 | 157 | ~~177~~ | 197 |
| ~~118~~ | ~~138~~ | ~~158~~ | ~~178~~ | ~~198~~ |
| ~~119~~ | 139 | ~~159~~ | 179 | 199 |
| ~~120~~ | ~~140~~ | ~~160~~ | ~~180~~ | ~~200~~ |

## Answers to Exercises 5-1

1. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

2. (a) 25 (b) 15

3. All of the numbers would be crossed out.

4. 10, 20, 30, 40, 50, 60, 70, 80, 90, 100.

5. 7, 14, 21, 28, 35, 42, 49.

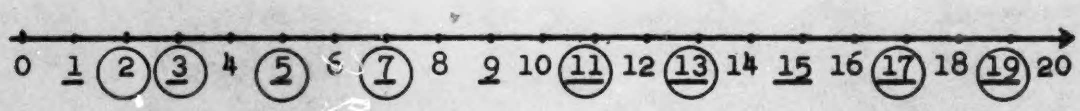
6. 33, 36, 39, 42, 45, 48, 51, 54, 57.

7. 15, 30, 45, 60, 75, 90.

8. (a) 8 (b) 3 and 5, 5 and 7, 11 and 13, 17 and 19, 29 and 31, 41 and 43, 59 and 61, 71 and 73.

9. Yes 3, 5, 7.

10. (a), (b), (c)



The other prime numbers between 1 and 50 are 21, 23, 29, 31, 37, 41, 43, and 47. (d) Yes.

11. 3, 5, 7, 11, 13, 17, 19, 23, 29

*12. (a) 8 is a multiple of 2

(b) 9 is a multiple of 3

(c) $7^2$ is 49, but $11^2$ is 121.

The questions following Exercise 12 are not intended as an assignment for all pupils. However, some pupils may wish to continue the sieve for numbers from 101 to 200. Such work should be entirely voluntary.

There are 21 prime numbers in the set of numbers from 101 to 200. The primes are 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199.

Twin primes are 101 and 103; 107 and 109; 137 and 139; 149 and 151; 179 and 181; 191 and 193; 197 and 199.

There are no prime triplets except 3, 5, 7 because every third odd number is a multiple of 3.

## 5-2. Factors

The purposes of this section are to develop understandings of the terms "factor," "complete factorization," "composite number," and of the Unique Factorization Property. Notice that the definition of factor includes zero, thus zero has factors. Exercise 4 of Exercises 5-2a serves as a basis of a discussion of zero.

We do not consider it mandatory that pupils use exponents. However, they should be encouraged to use them wherever it is reasonable to do so.

As a matter of interest, the Unique Factorization Property is usually referred to as the Fundamental Theorem of Arithmetic.

Notice that in complete factorization for a prime number like 17 only one factor is written, as: 17 = 17.

## Answers to Exercises 5-2a

1. (a) 1, 2, 5, 10
   (b) 1, 3, 5, 15
   (c) 1, 3, 9
   (d) 1, 2, 3, 6, 9, 18
   (e) 1, 3, 9, 27
   (f) 1, 2, 3, 4, 6, 8, 12, 24
   (g) 1, 11
2. (a) $1 \cdot 10$; $2 \cdot 5$
   (b) $1 \cdot 15$; $3 \cdot 5$
   (c) $1 \cdot 9$; $3 \cdot 3$
   (d) $1 \cdot 100$; $2 \cdot 50$; $4 \cdot 25$; $5 \cdot 20$; $10 \cdot 10$
   (e) $1 \cdot 24$; $2 \cdot 12$; $3 \cdot 8$; $4 \cdot 6$
   (f) $1 \cdot 16$; $2 \cdot 8$; $4 \cdot 4$
   (g) $1 \cdot 72$; $2 \cdot 36$; $3 \cdot 24$; $4 \cdot 18$; $6 \cdot 12$; $8 \cdot 9$
   (h) $1 \cdot 81$; $3 \cdot 27$; $9 \cdot 9$
3. (a) $2 \cdot 5$
   (b) $3 \cdot 5$
   (c) $3 \cdot 3$ or $3^2$
   (d) $2 \cdot 3 \cdot 5$
   (e) $3 \cdot 3 \cdot 5$ or $3^2 \cdot 5$
   (f) $2 \cdot 5 \cdot 5$ or $2 \cdot 5^2$
   (g) 13

4. Zero is not a factor of six since there is not a number which when multiplied by zero which gives a product of six; 6 is a factor of zero since the product of six and zero is zero, thus the definition is satisfied.

5. 1 and 20; 1 and 72

6. (a) $3 \cdot 5 \cdot 7$

   (b) $2 \cdot 3 \cdot 7$

   (c) $3 \cdot 5 \cdot 5$ or $3 \cdot 5^2$

   (d) $3 \cdot 2 \cdot 2 \cdot 5 \cdot 5$ or $3 \cdot 2^2 \cdot 5^2$

   (e) $2^6$

   (f) $3 \cdot 5 \cdot 23$

   (g) 311 (This is a prime number)

   (h) $2^3 \cdot 5^3$

   (i) $7 \cdot 43$

   (j) $17 \cdot 19$

7. (a) Even

   (b) Even

   (c) Even

   (d) Even

   (e) Odd

   (g) Even

   (h) Odd

   (i) Odd

   (j) Odd

   (k) Even

8. (a) $11_{three}$ is four; it is even.

   (b) $12_{seven}$ is seven; it is odd.

   (c) $33_{five}$ is eighteen; it is even.

   (d) $101_{two}$ is five; this is odd.

9. Divisibility is a property of number. It is the number which is divisible by another number. The numeral is a way of writing the number. In base ten, a numeral which represents an

even number ends with an even number; in base five this is not necessarily so. This is illustrated in Exercise 8.

## Divisibility

The purpose of this assignment is to define divisibility through the Euclidean division algorithm. It does not seem important that seventh graders memorize the term "algorithm". We use the term to identify it in mathematical language and background.

Through use of the algorithm it is illustrated that the remainder becomes very important. It is well to emphasize the importance of the remainder since too many pupils regard it as trivial information. In Chapter 6, the remainder becomes important for rational numbers and in Chapter 12, modular arithmetic is an arithmetic of remainders.

The Euclidean division algorithm should be discussed in class. More than likely, most pupils will not be able to understand the formula without some interpretations. Such a question as: "What kind of number is $q$?" is valuable for class discussion. The symbol $b$ represents a counting number since the divisor cannot be zero for division by zero is undefined. This point should also be discussed in class.

## Answers to Exercises 5-2b

1. (a) If a number is divided by 11 and the remainder is zero, the number is divisible by 11.

   (b) The pupil should be encouraged to say this in his own words. If $a = 11q + R$ and $R = 0$, then $a$ is divisible by 11. ( $a$ is a whole number)

2. (a) If a = 1: 615
   a = 4: 645 All three divisible by three
   a = 7: 675
   (b) None
   (c) Set A. All the elements of the set.

3. (a) 216
   (b) None
   (c) x = 0; a and b can be any whole number.

4. (a) 5289 = 37(142) + 35. 5289 is not divisible by 37 since the remainder is not zero.
   (b) 4066 = 64(63) + 34. 4066 is not divisible by 64.
   (c) 30 must be added to 4066
   (d) 65003 = 397(163) + 292; add 105 to the number

5. (a) a = 13, R = 0
   (b) a = 28, R = 0
   (c) a = 8, R = 4
   (d) a = 6, R = 5

6. (a) Any number for a other than 13.
   (b) There is no limit to the replacements that could be used for a to make the sentence false.

## 5-3. Diagrams of Numbers

This section is intended to increase the pupils' understanding of "factor" and to introduce composite numbers in a geometric setting. This is just another way of thinking about numbers.

In making a diagram you may suggest to the pupils that it is best first to make a list of all the factors of the number and to arrange them from the smallest to the largest. For 12 this would be 1, 2, 3, 4, 6, 12. Then start with 1 and continue to build a chain so that the second number is divisible by the preceding number, and so on. Thus, one chain would consist of 1, 2, 4, and 12; another chain would consist of 1, 3, 6, and 12; and the last chain would be 1, 2, 6, and 12. In each of these chains, taking any pair, the second is divisible by the first and there is no factor between them. We could not go from 1 to 4 since there is the factor 2 between 1 and 4. Remember that one of the rules of the game is that there may be no other factor between successive numbers.

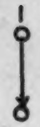
We see no point in expecting seventh grade pupils to master these diagrams. They are merely introduced here to illustrate another way of "looking at" numbers. Since some pupils like to make drawings this may help them identify kinds of numbers by geometric patterns.

## Answers to Exercises 5-3a

1. (a) 35 (b) 75 (c) 25

2. (a) a = 2 (b) a = 14 (c) a = 6
b = 2 b = 49 b = 12

3.



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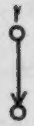

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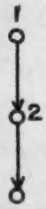

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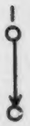

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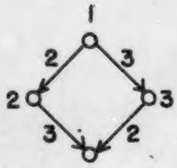

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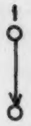

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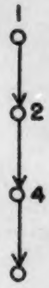

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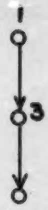

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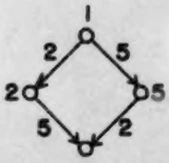

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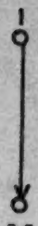

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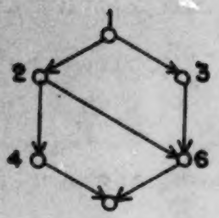

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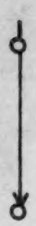

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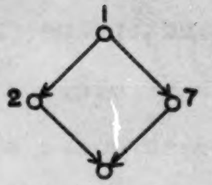

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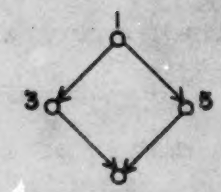

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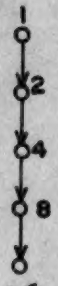

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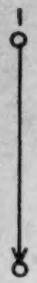

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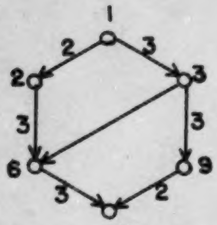
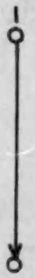

1
2
3
2
3
3
3
6
9
3
2

18



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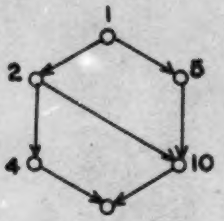


20

*4. (a) 2, 3, 5, 7, 11, 13, 17, and 19

(b) 4, 9

(c) 6, 10, 14, 15

(d) 8

(e) 12, 18, 20

*5. (a) ab (b) $a^2$ (c) ab (d) ab; $a^3b$

*6. (a) 21, 22, 26, 33, 34, and so on.

(b) 25, 49, 121 or any other square of a prime number.

(c) 28, 44, 45 or any others of the form $a^2b$ where a and b are prime.

(d) 24, 40, 54, 135 or any others of the form $a^3b$.

(e) Here are some possible forms: $a^2b^2$: 36, 100, 225; $a^3b^2$: 72, 200, 108

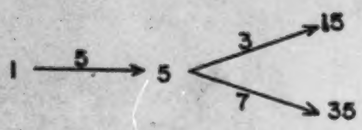
## Sketching the Distributive Property

The sketch introduced in this section is commonly called a "tree diagram". Naturally tree diagrams would illustrate patterns even if the numbers were not prime numbers. Notice that Example 4 has a "1" as one of its arrows.

As a matter of interest quite often tree diagrams are used in probability and game theory as a means of describing a problem.

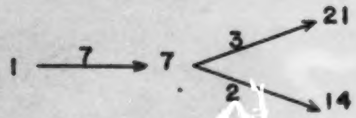
## Answers to Exercises 5-3b

1.



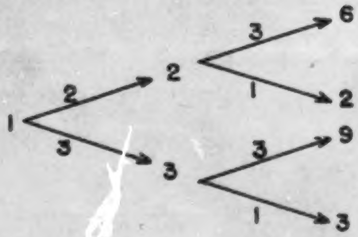
15 + 35 = 50

2.



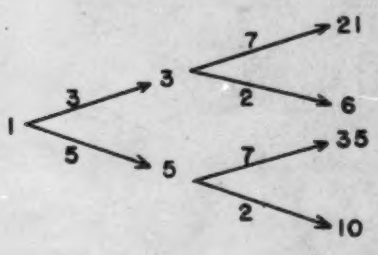
21 + 14 = 35

3.


1
2
2
3
6
1
2
3
3
3
9
1
3

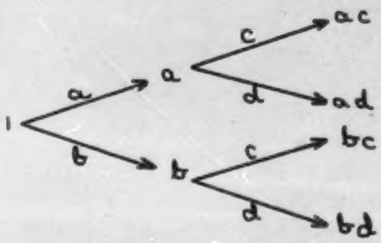
6 + 2 + 9 + 3 = 20

4.


1
3
3
7
21
2
6
5
5
7
35
2
10

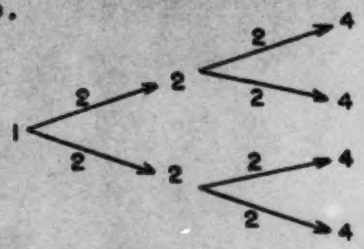
21 + 6 + 35 + 10 = 72

5.


1
a
a
c
ac
d
ad
b
b
c
bc
d
bd

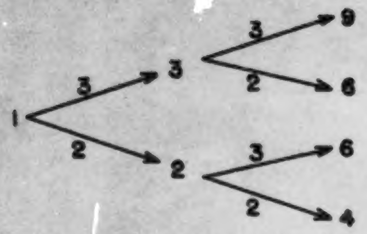
ac + ad + bc + bd

6.


1
2
2
2
2
2
2
2
2
2
2
4
4
4
4

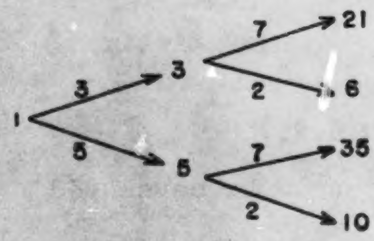
4 + 4 + 4 + 4 = 16

7.


1
3
2
3
2
3
2
3
2
9
6
6
4

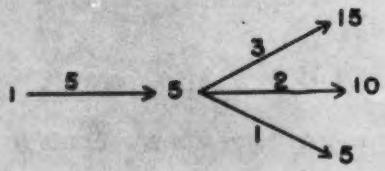
9 + 6 + 6 + 4 = 25

8.


1
3
5
3
5
7
2
7
2
21
6
35
10

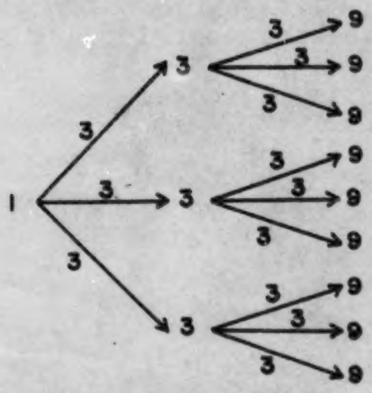
$a^2 + ab + ab + b^2$ or $a^2 + 2ab + b^2$

9.


1
5
5
3
2
1
15
10
5

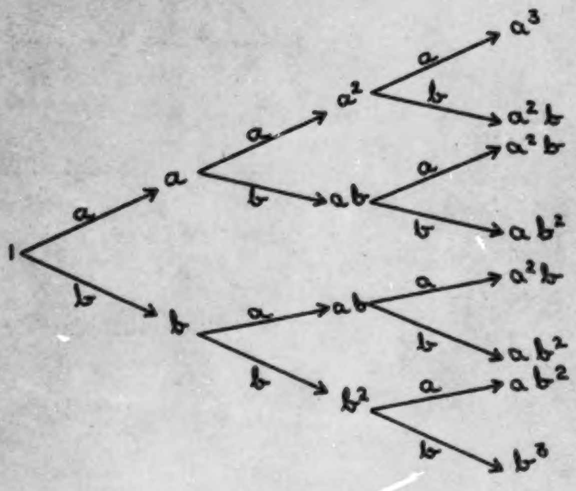
15 + 10 + 5 = 30

10.


1
3
3
3
3
3
3
3
3
3
3
3
3
3
3
3
9
9
9
9
9
9
9
9
9

The sum is 81.

11. There is one path (or branch) to 8; three to 4; three to 2; and one to 1.
12. Compare the coefficients in this exercise to the answers of Exercise 11 above.



$$a^3 + a^2b + a^2b + ab^2 + a^2b + ab^2 + ab^2 + b^3$$

$$a^3 + 3a^2b + 3ab^2 + b^3$$

## 5-4. Greatest Common Factor (G.C.F.)

The following skills and understandings should be developed through this section:

1. To identify the common factors of several numbers.
2. To find the greatest common factor of several numbers in two ways--by listing the factors of each and by using the complete factorization of each number.

It is recommemded that the abbreviation (G.C.F.) should not be used in this section. Use the complete phrase "greatest common factor" in order to fix the meaning of the concept in the minds of the pupils.

Explain that if something is "common" to several things, then one meaning is that they are similar in some way. If two numbers have a certain number as a factor, we say that factor is common to both.

Problem *10 could be suggested for overnight thought by the students, with a discussion of their conclusions the next day.

Provide drill on finding the greatest common factor by using complete factorizations. This is the "shorter method" that the pupils are asked to discover; it is illustrated and discussed below.

Be sure the pupils use short division when the divisor is less than ten.

Method of finding the Greatest Common Factor by Complete Factorization. This is a procedure for computing the greatest common factor but it is not intuitive. The method of the definition (listing all the factors, etc.) should be used in class until the concept is fixed in the minds of the pupils.

Illustration of Method: Find the greatest common factor of 24 and 60.

First, find the complete factorization of each number:

$24 = 2^3 \times 3$

$60 = 2^2 \times 3 \times 5.$

The factor 2 occurs three times in the complete factorization of 24 and twice for 60; hence, 2 can occur at most twice in any factor common to both 24 and 60. The factor 3 occurs just once in each, and hence can occur at most once in any common factor. The number 5 is not a factor of 24, and hence cannot occur in any common factor. Hence the greatest common factor is $2^2 \times 3 = 12$. Notice that the power of each prime factor of the greatest common factor is the smaller of the powers to which it occurs in the two given numbers.

Summary of Method: (1) Find the complete factorization of each number. (2) Pick out the primes which occur in all (every one) of these factorizations. (3) For each prime picked in (2), write the smallest power of it which occurs in any of the factorizations. (4) Multiply all the powers written in (3); this product is the greatest common factor.

The pupils will find it easier to pick out the primes and exponents in steps (2) and (3) above if the original numbers are written in a column with the complete factorization of each number to the right of the number.

## Discussion of Exercises

4. This exercise gives the pupils an opportunity to establish a procedure for searching for factors of numbers when they cannot guess them. It is time consuming, and may be omitted if time is not available for it.

   (a) In factoring 1173, remind the pupils of the test for divisibility by 3 and obtain 1173 = 3 × 391. For the number 391, we try the successive primes 2, 3, 5, 7, 11, 13, 17, to find 391 = 17 × 23. Review the reasons for trying only the primes--i.e. for skipping 9, 15, and all the even numbers except 2. Point out that, since $20 \cdot 20 = 400$ is larger than 391, it is only necessary to try the primes less than 20 (391 cannot be the product of two factors each of which is larger than 20). In general, if the primes are tried in order of magnitude, one should stop when the quotient is less than or equal to the divisor.

9. Since a is a prime, the only factors of a are 1 and a. If a is also a factor of b, it is the greatest common factor; if a is not a factor of b, the greatest (and only) common factor is 1. Illustrate with $a = 7$, $b = 49$ and $a = 7$, $b = 48$.

*10. The following proof is for the information of the teacher. It is not expected that any except the most brilliant students would profit by it in the 7th grade.

To show there is no largest prime number, we will show that if p is any prime, there is another prime larger than p. Denote by M the product of all the primes less than or equal to p : $M = 2 \times 3 \times 5 \times 7 \times 11 \times \ldots \times p$.
Then $M + 1$ is certainly larger than p and $M + 1$ has at least one prime factor (it may be a prime). But $M + 1$ does not have any of the primes 2, 3, 5, 7, ..., p as a factor since division by any of these primes leaves a remainder of 1. Thus all the prime factors of $M + 1$ are larger than p and p is not the largest prime. Since p was an arbitrary prime, there is no largest prime.

12. BRAINBUSTER. Illustrate the results in (a), (b), (c) below using the values $a = 6$, $b = 9$, $c = 12$.

(a) The set of counting numbers is closed under the operation * since, for any counting numbers a and b, $a * b =$ the greatest common factor of a and b is again a counting number.

(b) The operation * is commutative since the greatest common factor of a and b is the same as the greatest common factor of b and a .

(c) The operation * is associative since each of $a * (b * c)$ and $(a * b) * c$ is the greatest common factor of the set $\{a, b, c\}$.

## Answers to Exercises 5-4

1. (a) $\{1, 2, 4, 8\}$ (d) $\{1, 19\}$
   (b) $\{1, 2, 3, 4, 6, 12\}$ (e) $\{1, 3, 7, 21\}$
   (c) $\{1, 2, 4, 8, 16\}$ (f) $\{1, 2, 4, 7, 14, 28\}$
2. (a) $\{1, 2, 4\}$ (d) $\{1, 2, 4\}$
   (b) $\{1, 3\}$ (e) $\{1, 2, 4\}$
   (c) $\{1\}$ (f) $\{1\}$
3. (a) $5 \times 7,\ 3 \times 7;\ 7$
   (b) $2^2,\ 2 \times 3,\ 2^3;\ 2$
   (c) $3 \times 7,\ 3^3,\ 3 \times 5;\ 3$
   (d) $2^2 \times 3 \times 5,\ 2 \times 3 \times 11,\ 2 \times 3 \times 7;\ 6$
   (e) $2^3 \times 3,\ 2^2 \times 3 \times 5,\ 2^2 \times 3 \times 7;\ 12$
   (f) $2 \times 3 \times 7,\ 3 \times 7^2,\ 3 \times 5 \times 7;\ 21$
   (g) $2 \times 3 \times 13,\ 3 \times 13;\ 13$
   (h) $2^2 \times 7,\ 2^3 \times 7,\ 2^4;\ 4$
   (i) $2^2 \times 3,\ 2 \times 3^2,\ 2^2 \times 5,\ 5 \times 7;\ 1$
   (j) $2 \times 5^3,\ 2^3 \times 5^2,\ 2 \times 5^2 \times 13;\ 50$
   (k) $2^2 \times 3 \times 5,\ 2^4 \times 3,\ 2^5 \times 3,\ 2^4 \times 3^2;\ 12$
   (l) $5,\ 5^2,\ 5^3;\ 5$
4. (a) $306 = 2 \times 3^2 \times 17$, $1173 = 3 \times 17 \times 23$,
   G.C.F. is $3 \times 17 = 51$. See discussion, Section 5-4.

(b) $2040 = 2^3 \times 3 \times 5 \times 17$, $2184 = 2^3 \times 3 \times 7 \times 13$,

G.C.F. is $2^3 \times 3 = 24$.

*(c) $15,912 = 2^3 \times 3^2 \times 13 \times 17$, $26,676 = 2^2 \times 3^3 \times 13 \times 19$,

$19,656 = 2^3 \times 3^3 \times 7 \times 13$, G.C.F. is $2^2 \times 3^2 \times 13 = 468$.

Part (c) is not difficult; it is starred because it is time consuming.

5. 6, 29, a

6. 1, 1, 1

7. Yes; e.g. 12, 15, 20

8. Yes; 1 is the least common factor of any two counting numbers.

9. If b is a multiple of a , the G.C.F. is a

If b is not a multiple of a , the G.C.F. is 1. See discussion, Section 5-4.

*10. No, there is no largest prime. See discussion, Section 5-4.

*11. 6. Notice that every whole number is a factor of 0 since $a \times 0 = 0$ for any whole number a .

12. BRAINBUSTER. See discussion, Section 5-4.

(a) Yes (b) Yes (c) Yes

## 5-5. Review

This section provides a review of the work of grades 1-6 and of chapters 1-4 of this book. It is placed here to allow the pupils an extra day to fix in their minds the concept of greatest common factor before being introduced to least common multiple.

Discussion of Exercises 5-5

4. (a) If we consider the experiment of drawing one piece of paper from the hat, there are 50 possible outcomes: the numeral drawn could represent any one of 1, 2, 3, ..., 50. Of

these 50 outcomes, there are 10 which are favorable (the numeral drawn represents a number divisible by 5), namely: 5, 10, 15, ..., 50. Thus the probability of a favorable outcome is 10/50 = 1/5.

(b) Of the 50 possible outcomes, there are 16 in which the numeral drawn represents a number which is divisible by 3, namely: 3, 6, 9, ... 48. To be sure of drawing at least one of these numerals, we would have to draw enough times to draw all the other numerals and then one more; i.e. (50 - 16) + 1 = 35 draws.

(c) If the numeral "50" is in base 7, there are $(35)_{ten}$ possible outcomes. Of these, 7 give numerals representing numbers divisible by 5 and the probability of drawing one of these is 7/35 = 1/5. There are $(11)_{ten}$ outcomes in which the numeral drawn represents a number divisible by 3; thus it would be necessary to draw (35 - 11) + 1 = 25 (base ten) times to be sure of drawing one of these 11 pieces of paper.

5. (a) Each of the sets {51, 53, 55, ..., 79} and {18, 20, 22, ..., 46} has 15 elements. There are many different ways in which a 1-1 correspondence can be set up between the two sets. Three of the correspondences are given below.

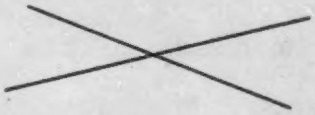
| | | |
|---|---|---|
| 51 ⟷ 18 | 51 ⟷ 46 | 51 ⟷ 20 |
| 53 ⟷ 20 | 53 ⟷ 44 | 53 ⟷ 22 |
| 55 ⟷ 22 | 55 ⟷ 42 | 55 ⟷ 24 |
| . . | . . | . . |
| . . | . . | . . |
| . . | . . | . . |
| 79 ⟷ 46 | 79 ⟷ 18 | 77 ⟷ 46 |
| | | 79 ⟷ 18 |

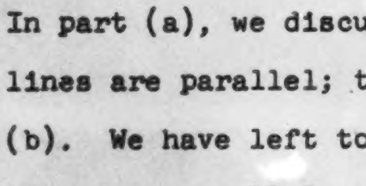
(b) Each of the sets $\{3, 6, 9, \ldots, 42\}$ and $\{105, 112, 119, \ldots, 196\}$ has 14 elements. There are many ways in which a 1-1 correspondence can be set up between the two sets.

*6. (a) Since $a$, $b$, $f$, $g$ are four different numbers and $g$ is a factor of both $a$ and $b$, we must have $g < a$, $g < b$. Also, $f$ is a factor of both $a$ and $b$ (different from $g$) and $g$ is the largest common factor of $a$ and $b$. Thus $f < a$, $f < b$ and $f < g$. Illustrate with $a = 12$, $b = 18$, $f = 3$, $g = 6$.

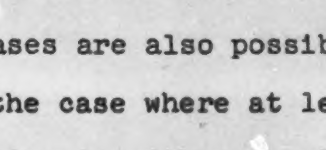
(b) If the numbers are allowed to be equal, all four of them may be the same so that there may be no inequalities among them. Illustrate with $a = b = f = g = 4$.

7. (a) Consider first two of the lines. Since they are different lines and are not parallel, they intersect in exactly one point as sketched. The third line either goes through this point of intersection or it doesn't. The two cases are illustrated below.



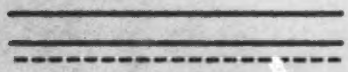


1 intersection

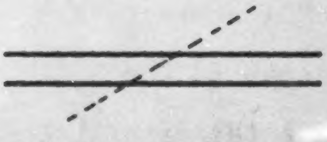


3 intersections

(b) In part (a), we discussed the cases where no two of the lines are parallel; these two cases are also possible in part (b). We have left to consider the case where at least two lines are parallel. The third line is either parallel to both of them or to neither of them. The two cases are illustrated below.



0 intersections



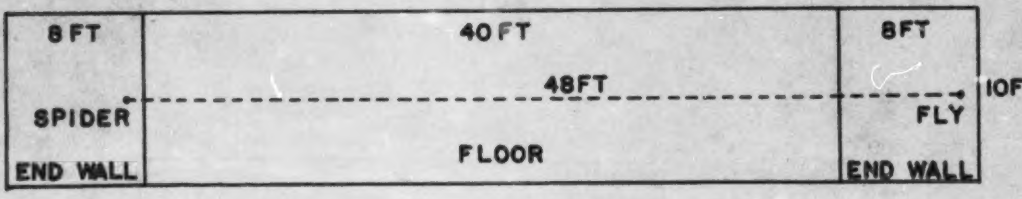
2 intersections

10. **BRAINBUSTER.** From the information in the problem, Mary got the 1st piece of candy and also the 49th (and possible others). Thus, if we start counting with Mary as number 1, and count each child as he or she takes a piece of candy, the child next to Mary will be counted as number 48. If we stop counting with 48, we will have counted all the children but, perhaps, we will have counted each child several times. In fact, the number of times we will have counted each child is exactly the same as the number of times the box of candy has been passed entirely around the table. This means that 48 must be a multiple of the number of children since, by counting each child the same (possibly multiple) number of times, the total count is 48. The children may wish to discuss whether they think one child (i.e. Mary sitting alone at a small table) satisfies the conditions.

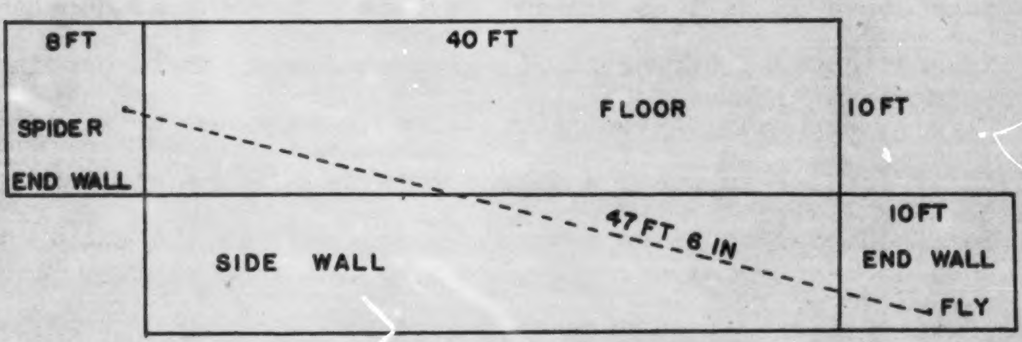
11. **SUPER BRAINBUSTER.** With the floor, walls, etc. in the positions they occupy in the room, it is hard to choose different possible routes for the spider and compare their lengths. Different routes are easier to find if we imagine the walls, etc. of the room laid down so that they lie in the plane of the floor. There are several such representations of the room (given below) and the shortest route associated with each representation is a

straight line segment. The lengths of these line segments can be measured on a scale drawing and compared.

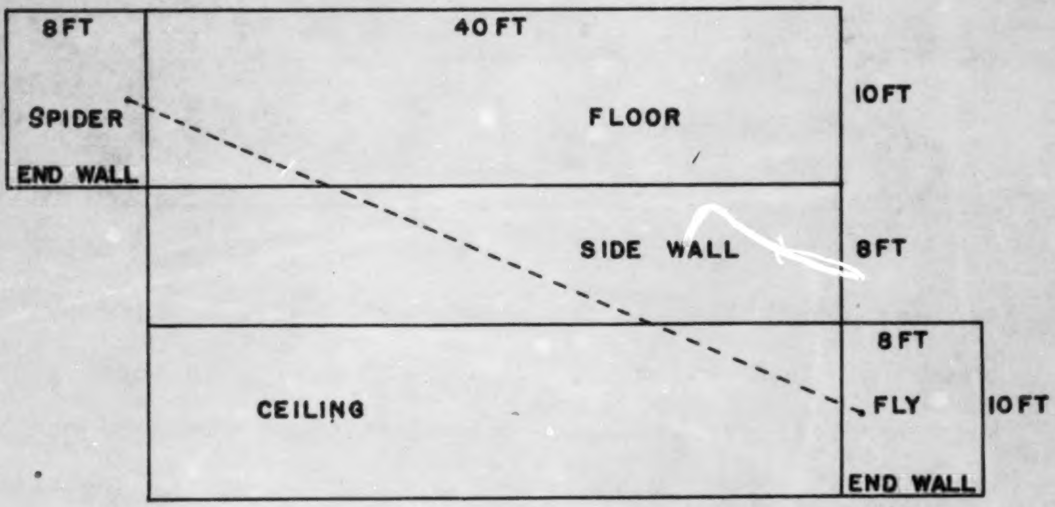
The easiest representation is to imagine the two end walls laid flat in the plane of the floor. This results in a route 48 feet long as shown on the sketch.


8 FT
SPIDER
END WALL
40 FT
48 FT
FLOOR
8 FT
FLY
END WALL
10 FT

Another possibility is to think of the end wall with the spider on it laid down as above, but the end wall with the fly on it is kept vertical at first and rotated into the plane of one of the side walls. Then that end and side wall together are laid flat in the plane of the floor. This results in a route about 47 feet 6 inches long as shown in the sketch.


8 FT
SPIDER
END WALL
40 FT
FLOOR
10 FT
SIDE WALL
47 FT 6 IN
10 FT
END WALL
FLY

Another possibility is to treat the end wall with the spider on it as before, but the end wall with the fly on it is first rotated up into the plane of the ceiling then that end wall and ceiling, and one side wall are laid down flat in the plane of the floor. (The ceiling is not just lowered; it is cut loose from one side wall and rotated with the other sidewall.) This results in a route about 45 feet 8 inches long as shown in the sketch.



This is the shortest path. The students should construct a scale model of the room from cardboard and try the representations above as well as others (perhaps different students will have discovered different representations on their own initiative). Accuracy to the nearest inch should not be expected of their models; in fact, a little care is required in constructing the model to be sure which measurement is the shortest.

## Answers to Exercises 5-5

Remember: Base SEVEN in exercises 1-3.

1. (a) 55 (b) 1101 (c) 1044 (d) 220,055

2. (a) 1, 3, 5, 21

(b) 1, 5, 34

(c) 1, 2, 4, 11, 32, 64, 161, 352; Note: in base ten, these numbers are 1, 2, 4, 8, 23, 46, 92, 184.

3. (a) 5 (b) 64

4. (a) 1/5 (b) 35 (c) 1/5, 25 (base ten) or 34 (base seven).

See discussion, Section 5-5.

5. (a) Yes (b) Yes. See discussion, Section 5-5.

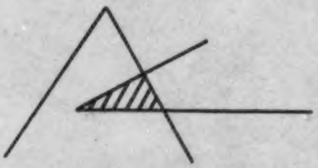
*6. (a) $f < g$, $f < a$, $f < b$, $g < a$, $g < b$

(b) None. See discussion, Section 5-5.

7. (a) 1 or 3 (b) 0, 1, 2, 3 See discussion, Section 5-5.

8. One example is: the front wall, the ceiling, and the left side wall. These three planes intersect in the upper left hand corner of the front wall. In most classrooms there are 8 different examples of sets of 3 planes intersecting in a point--one example for each of the 8 corner points of the room.

9. Many different examples are possible. One sketch is shown here.



10. **BRAINBUSTER.** 48, 24, 16, 12, 8, 6, 4, 3, 2, 1. See discussion, Section 5-5.

11. **SUPER BRAINBUSTER.** The shortest path for the spider to follow passes diagonally down the end wall, diagonally across one corner of the floor, diagonally up the side wall, diagonally across one corner of the ceiling and diagonally down the other end wall to the fly. The length of this path is about 45 feet 8 inches. See discussion, Section 5-5.

12. (a) seven hundred forty thousand three
    (b) seven hundred thousand four hundred three
    (c) seven hundred thousand forty-three
    (d) seven hundred four thousand three
13. (a) 801 (b) 307,000 (c) 21,024 (d) 4,500,000
14. 7932
15. 20,394
16. 5077
17. 2,229,793,830
18. 1423
19. 903
20. 1660 1/5
21. 364
22. 607 1/23
23. 3 1/5 or 16/5
24. 20
25. 5/6
26. 11
27. 3 1/6
28. 12 5/6
29. 9
30. 0
31. 1

## 5-6. Least Common Multiple (L.C.M.)

Skills and Understandings to be developed:

1. To find common multiples of several numbers.
2. To find the least common multiple of several numbers in

two ways--by listing the multiples of each and by using the complete factorization of each number.

It is recommended that the abbreviation (L.C.M.) should not be used in this section. Use the complete phrase "least common multiple" in order to fix the meaning of the concept in the minds of the pupils.

Stress that "multiple" in this sense means "multiple by a counting number"--zero is not acceptable as a multiple in this sense. (If zero were allowed as a multiplier, then every number would have zero as its least multiple, since $0 \times a = 0$ for any number $a$.) Thus zero is not a multiple of 6 even though $0 \times 6 = 0$ (see Exercise 10).

Exercise *11 could be suggested for overnight thought by the students, with a discussion of their conclusions the next day.

Provide drill on finding the least common multiple by using complete factorizations. This is the "shorter method" that the pupils are asked to discover. It is a procedure for computing the least common multiple but it is not intuitive. The method of the definition (listing the multiples, etc.) should be used until the concept is fixed in the minds of the pupils.

Example: Find the least common multiple of 12 and 18.

First, find the complete factorization of each number:

$$12 = 2^2 \times 3$$
$$18 = 2 \times 3^2.$$

Any number which is a multiple of 12 must have $2^2$ as a factor and also 3 as a factor. Any number which is a multiple of 18 must have 2 as a factor and also $3^2$ as a factor. Thus a number which is a common multiple of 12 and 18 must have among its factors all of the following:

$$2^2, \quad 3, \quad 2, \quad 3^2.$$

But a number which has $2^2$ as a factor certainly has 2 as a factor. Similarly, a number which has $3^2$ as a factor certainly has 3 as a factor. Thus any number which has the two factors $2^2$ and $3^2$ will have all four of the factors $2^2$, 3, 2, $3^2$. The smallest number which has the factors $2^2$, and $3^2$ is $2^2 \times 3^2 = 36$.

On the other hand, $2^2 \times 3^2$ is a multiple of 12 (which is $2^2 \times 3$) and it is also a multiple of 18 (which is $2 \times 3^2$). It is their least common multiple. Notice that the power of each prime factor in the least common multiple is the larger of the powers to which it occurs in the two given numbers.

Summary of Method: (1) Find the complete factorization of each number. (2) Notice which primes occur in at least one of the factorizations. (3) For each prime noticed in (2), write the largest power of it which occurs in any of the factorizations. (4) Multiply all the powers written in (3); this product is the least common multiple.

The pupils will find it easier to pick out the primes and exponents in steps (2) and (3) above if the original numbers are written in a column with the complete factorization of each number to the right of the number.

Discussion of Exercises 5-6

7. Addition and subtraction of fractions have not yet been discussed in this text; the students will be using their 6th grade skills. The amount 1/12 cup can be obtained by first filling the 1/3 cup

measure and then removing from it 1/4 of a cup, leaving 1/3 - 1/4 = 1/12 cup. Multiples of 1/12 cup can be obtained by repeating this operation (for many of them, e.g. 4/12 = 1/3, a simpler procedure is available.)

*9. If $b$ is a multiple of $a$, then it is the least common multiple of $a$ and $b$ since it is the smallest multiple of $b$. If $b$ is not a multiple of $a$, then the prime $a$ does not appear in the complete factorization of $b$ (unique factorization property). Thus, in forming the least common multiple of $a$ and $b$ by taking the highest power of each prime which occurs in the complete factorization of either $a$ or $b$, we would take $a^1$ (since it is the only prime factor of $a$ and does not occur in the factorization of $b$), and we would take each of the prime-power factors in the complete factorization of $b$ (since each of these primes is different from $a$). This gives the least common multiple equal to $a \times b$. Illustrate with $a = 7$, $b = 49$ and $a = 7$, $b = 48$.

10. The word "multiple" refers to multiplication by counting numbers only. Thus the only multiple of 0 is 0, and every multiple of 6 is at least as big as $1 \times 6 = 6$. There are no common multiples of 0 and 6, and hence, no least common multiple.

*11. The least common multiple of two numbers is at least as big as the larger of the two numbers (since the L.C.M. is a multiple of the larger number). The greatest common factor of two numbers is no larger than the smaller of the two numbers (since the G.C.F. is a factor of the smaller number.) If the least common multiple and greatest common factor are equal, the larger and smaller number must also be equal. Illustrate with 4 and 6, locating the L.C.M. and G.C.F. on the number line; then use 4 and 4.

12. **BRAINBUSTER.** Illustrate the results in (a), (b), (c) below using a = 4, b = 6, c = 9.

    (a) The set of counting numbers is closed under the operation o since, for any counting numbers a and b, a o b = the least common multiple of a and b is again a counting number.

    (b) The operation o is commutative since the least common multiple of a and b is the same as the least common multiple of b and a.

    (c) The operation o is associative since each of a o (b o c) and (a o b) o c is the least common multiple of the set of numbers {a, b, c}.

## Answers to Exercises 5-6

1. (a) {6, 12, 18, 24, 36, 42, 48, 54, 60, 66, 72, 78, 84, 90, 96}
   (b) {8, 16, 24, 32, 40, 48, 56, 64, 72, 80, 88, 96}
   (c) {9, 18, 27, 36, 45, 54, 63, 72, 81, 90, 99}
   (d) {12, 24, 36, 48, 60, 72, 84, 96}
2. (a) {24, 48, 72, 96}
   (b) {12, 24, 36, 48, 60, 72, 84, 96}
   (c) {24, 48, 72, 96}
   (d) {72}
3. (a) 10 (d) 12
   (b) 6 (e) 72
   (c) 12 (f) 360

(g) 210 (j) 80

(h) 660 (k) 36

(i) 210 (l) 156

4. (a) 1,000 *(b) 6,480 *(c) 7,038

Parts (b) and (c) are not difficult; they are starred because they are time consuming.

5. 6, 29, a

6. 6, 29, a

7. 1/12, 2/12 = 1/6, 3/12 = 1/4, 4/12 = 1/3, 5/12, 6/12 = 1/2, 7/12, 8/12 = 2/3, 9/12 = 3/4, 10/12 = 5/6, 11/12. See discussion of Exercises 5-6.

8. No, there is no largest common multiple of 6 and 9 (or of any two counting numbers), since, if m is any multiple, then 2 × m is a larger one.

*9. If b is a multiple of a, the L.C.M. is b. If b is not a multiple of a, the L.C.M. is a × b. See Discussion of Exercises 5-6.

10. There are no common multiples of 0 and 6; hence there is no least common multiple. See Discussion of Exercises 5-6.

*11. The two numbers must be equal. See Discussion of Exercises 5-6.

12. BRAINBUSTER. See Discussion of Exercises 5-6.

(a) Yes (b) Yes (c) Yes

## 5-7. Problems with G.C.F. and L.C.M.

### Exercises 5-7

The purpose of this section is to fix in the pupil's mind the difference between the greatest common factor and the least common multiple. The discussion should emphasize again that the greatest common factor of two different numbers is smaller than one or both of the numbers. Also, the least common multiple of two different numbers is greater than one or both of the numbers.

## Answers to Exercises 5-7

1. 2 × 2 × 2; 2 × 3; 2; 2 × 2 × 2 × 3 or 24
2. 3 × 5; 5 × 5; 5; 3 × 5 × 5 or 75
3. 2 × 2 × 3 × 3; 2 × 2 × 3; 2 × 2 × 3 or 12; 2 × 2 × 3 × 3 or 36
4. 3 × 7; 3 × 5; 3; 3 × 7 × 5 or 105
5. 23; 43; 1; 23 × 43 or 989
6. 2 × 3 × 11; 2 × 3 × 13; 2 × 3 or 6; 2 × 3 × 11 × 13 or 858
7. 3 × 13; 3 × 17; 3; 3 × 13 × 17 or 663
8. 2 × 37; 2 × 73; 2; 2 × 37 × 73 or 5402
9. 3 × 3 × 5; 2 × 2 × 2 × 3 × 3; 2 × 2 × 2 × 3 × 3 × 5 × 7 or 2520

   2 × 2 × 3 × 3 × 7; 3 × 3 or 9
10. 2 × 2 × 11; 2 × 2 × 23; 2 × 2 × 11 × 31 or 31,372;

    2 × 2 × 31; 2 × 2 or 4
11.

| (a) | (b) |
|---|---|
| 8 × 6 = 48 | 2 × 24 = 48 |
| 15 × 25 = 375 | 5 × 75 = 375 |
| 36 × 12 = 432 | 12 × 36 = 432 |
| 21 × 15 = 315 | 3 × 105 = 315 |
| 23 × 43 = 989 | 1 × 989 = 989 |
| 66 × 78 = 5148 | 6 × 858 = 5148 |
| 39 × 51 = 1989 | 3 × 663 = 1989 |
| 74 × 146 = 10,804 | 2 × 5402 = 10,804 |

12. 6, 15, 24, 33, 42

    11, 22, 33, 44, 55 L.C.M. is 33
13. 110 is 10 × 11

    1000 is 10 × 10 × 10 G.C.F. is 10

14. (a) 3, 6, 9, 12, 15--first bell

5, 10, 15--second bell

They strike together again in 15 minutes

(b) 6, 12, 18, 24, 30--first bell

15, 30--second bell

They strike together again in 30 minutes

(c) It is the L.C.M. of the times

15. (a)

| Rows | Bulbs per row |
|---|---|
| 1 | 112 |
| 2 | 56 |
| 4 | 28 |
| 8 | 14 |
| 16 | 7 |

(Bulbs and rows may be interchanged.)

16. It is the product of the two numbers.

17. No. e.g. 6 2 × 3, 15 3 × 5, 10 2 × 5

G.C.F. is 1; L.C.M. is 2 × 3 × 5 or 30

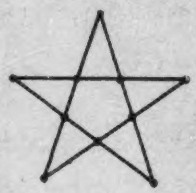
18. (a) 1. One is the least common factor of any two numbers.

(b) No answer is possible for the greatest common multiple of any two numbers.

19. No. It is not possible to have exactly four numbers between two odd numbers.

20. Yes. e.g. 24, 25, 26, 27, 28 are composite numbers between 23 and 29, which are prime.

21. BRAINBUSTER. The pattern is a five-pointed star.



## Sample Test Questions for Chapter 5

I. True-False Questions

(T) 1. Every composite number can be factored into prime numbers in exactly one way, except for order.

(F) 2. The sum of an odd and an even number is always an even number.

(T) 3. Some odd numbers are *not* primes.

(F) 4. Every composite number has *only* two prime factors.

(F) 5. The number 51 is a prime.

(T) 6. All even numbers have the factor 2.

(F) 7. Any multiple of a prime number is a prime.

(T) 8. 8 can be expressed as the sum of twin primes.

(T) 9. Even though 1 has as factors only itself and 1, it is *not* considered a prime number.

(F) 10. All odd numbers have the factor 3.

(F) 11. No even number is a prime.

(T) 12. The least common multiple of 3, 4, and 12 is 12.

(F) 13. The number one is a prime factor of all counting numbers.

(F) 14. The greatest common multiple of 2, 5 and 10 is 100.

(T) 15. The difference between any two prime numbers greater than 100 is always an even number.

(F) 16. The number 41 is a composite number.

(F) 17. A multiple of 6 must be a multiple of 18.

(F) 18. The least common multiple of 2 and 6 is 12.

(T) 19. The greatest common factor of any two even numbers is at least 2.

## II. *Multiple Choice*

20. Which of the following is an odd number?

(a) $(24501)_{\text{six}}$

(b) $(60654)_{\text{eight}}$

(c) $(TE104)_{\text{twelve}}$

(d) $(110100)_{two}$

(e) None of the above 20 (a)

21. The greatest common factor of 48 and 60 is:

(a) 2 × 3
(b) 2 × 2 × 3
(c) 2 × 2 × 2 × 2 × 3 × 5
(d) 2 × 2 × 2 × 2 × 2 × 2 × 3 × 3 × 5
(e) None of the above 21 (b)

22. Every counting number has at least the following factors:

(a) Zero and one
(b) Zero and itself
(c) One and itself
(d) Itself and two
(e) None of the above 22 (c)

23. In the complete factorization of a number

(a) All the factors are primes.
(b) All the factors are composites.
(c) All the factors are composite except for the factor 1.
(d) All the factors are prime except for the factor 1.
(e) None of the above 23 (a)

24. How many different prime factors does the number 72 have?

(a) 0
(b) 1
(c) 2
(d) 3
(e) None of the above 24 (c)

25. The least common multiple of 8, 12, and 20 is:

(a) 2 × 2
(b) 2 × 3 × 5
(c) 2 × 2 × 2 × 3 × 5
(d) 2 × 2 × 2 × 2 × 2 × 2 × 3 × 5
(e) None of the above 25 (c)

26. Which of the following is an even number?

(a) $(100)_{three}$

(b) $(100)_{five}$

(c) $(100)_{seven}$

(d) $(100)_{\text{eleven}}$
(e) None of the above 26 (e)

27. Which of the following numbers is odd?

(a) 17 × 18
(b) 18 × 11
(c) 11 × 20
(d) 99 × 77
(e) None of the above 27 (d)

28. Which of the following is not a prime number? There is only one.

(a) 271
(b) 277
(c) 281
(d) 282
(e) 283 28 (d)

29. Let a represent an odd number, and b represent an even number; then a + b must represent

(a) an even number.
(b) a prime number.
(c) an odd number.
(d) a composite number.
(e) none of these. 29 (c)

30. If n represents an odd number, the next odd number can be represented by

(a) n + 1
(b) n + 2
(c) n + 3
(d) 2 × n
(e) None of these 30 (b)

31. A counting number is an even number if it has the factor:

(a) 5
(b) 3
(c) 2
(d) 1
(e) None of these 31 (c)

32. Which of the following sets contains only even numbers?

(a) {2, 3, 4, 5, 9, 10}
(b) {2, 5, 10}
(c) {3, 5, 9}
(d) {2, 4, 10}
(e) {3, 9} 32 (d)

33. Which of the following is a prime number?

(a) 4
(b) 7
(c) 9
(d) 33
(e) None of these

33 (b)

34. Which of the following is not a factor of 24?

(a) 2
(b) 3
(c) 4
(d) 9
(e) 12

34 (d)

35. Which of the following is the complete factorization of 36?

(a) 4 × 9
(b) 2 × 3 × 6
(c) 3 × 12
(d) 2 × 18
(e) 2 × 2 × 3 × 3

35 (e)

36. The numbers 8, 9, 16, 20, 27, and 72 are all

(a) prime numbers.
(b) even numbers.
(c) odd numbers.
(d) composite numbers.
(e) none of these.

36 (d)

37. How many multiples of 4 are there between 25 and 50?

(a) 5
(b) 7
(c) 9
(d) 11
(e) None of these

37 (e)

38. If a whole number has 6 as a factor, then it also has the following factors:

(a) 2 and 3
(b) 2 + 3
(c) 12
(d) all multiples of 6
(e) none of these

38 (a)

39. Suppose p and q are counting numbers and q is a factor of p; then:

(a) q is a multiple of p
(b) p is a multiple of q
(c) q must be a prime number
(d) the greatest common factor of p and q must be less than q
(e) none of these

39 (b)

40. The greatest common factor of 60 and 42 is

(a) 2 × 3
(b) 2 × 2 × 3
(c) 2 × 3 × 5
(d) 2 × 3 × 7
(e) none of these

40 (a)

41. The greatest common factor of two different numbers must be:

(a) a composite number
(b) a prime number
(c) smaller than one of the numbers
(d) smaller than both of the numbers
(e) none of these

41 (c)

42. The least common multiple of two numbers is always:

(a) their product
(b) the product of their factors
(c) the sum of their factors
(d) the sum of the numbers
(e) none of these

42 (e)

43. Which of the following statements describes a prime number?

(a) a number which is a factor of a counting number
(b) a number which has no factors
(c) a number which does not have 2 as a factor
(d) a number which has exactly 2 different factors
(e) none of these

43 (d)

44. How many prime numbers are there between 20 and 40?

(a) 4
(b) 5
(c) 8
(d) 9
(e) none of these

44 (a)

45. When two prime numbers are added the sum is

(a) always an odd number

(b) always an even number
(c) always a composite number
(d) always a prime number
(e) none of these 45 (e)

46. The set of factors of the number 12 is

(a) {1, 2, 3, 4, 8, 12}
(b) {1, 2, 3, 4, 6, 12}
(c) {1, 2, 3, 4, 6}
(d) {2, 3, 4, 6, 12}
(e) none of these 46 (b)

47. How many different factorizations of two factors each does 175 have?

(a) 2
(b) 3
(c) 4
(d) 5
(e) none of these 47 (b)

48. The number of factors in the complete factorization of 182 is

(a) 2
(b) 3
(c) 4
(d) 5
(e) none of these 48 (b)

| III. Problems | Answers |
|---|---|
| 49. Find the complete factorization of each number | |
| (a) 16 | $2^4$ |
| (b) 100 | $2^2 \times 5^2$ |
| (c) 57 | $3 \times 19$ |
| 50. Find the greatest common factor of each set of numbers | |
| (a) 5 and 25 | 5 |
| (b) 18 and 27 | 9 |
| (c) 60, 36 and 24 | 12 |
| 51. Find the least common multiple of each set of numbers. | |
| (a) 6 and 8 | 24 |
| (b) 7 and 9 | 63 |
| (c) 16, 12 and 20 | 240 |

52. Find the smallest number which has a factorization composed of 3 composite numbers. $64 = 4^3$

53. Show that a product is even if one (or more) of its factors is even.

54. Is the set of even numbers closed under addition? Show that your answer is correct. Yes

55. If the complete factorization of a number is 2 × 3 × 5 × 7, what factors less than 20 does the number have?
1, 2, 3, 5, 6, 7, 10, 14, 15.

56. Find all the common multiples less than 100 of these three numbers: 3, 6, 9.
18, 36, 54, 72, 90.

57. Write all the factorizations of two factors for the number 210.
1 × 210, 2 × 105, 3 × 70, 5 × 42,
6 × 35, 7 × 30, 10 × 21, 14 × 15.

## CHAPTER 6

# THE RATIONAL NUMBER SYSTEM I

### Overview

The most important objective of this chapter is to present the reasons behind the rules for the arithmetic operations of multiplication, division, addition, and subtraction on the set of rational numbers. The student will be able to use these rules correctly and with facility if he fully understands the reasons behind them. We reiterate the principle that mathematics stresses reason over rote learning and concept over technique. At the same time, an equally important goal at this level is to reinforce techniques which the pupil has already learned. These two goals should not be incompatible. We are assuming that the pupils have already been introduced to these operations and have had some practice in them in an earlier grade. This means that they already know what a fraction is--at least in easy situations.

The seventh grader is probably not ready for a postulational approach to the study of number systems. The whole numbers and the rational numbers are part of nature for him. To speak of _constructing_ the rational number system from the whole numbers would be meaningless at his level, as it would be to speak of the problem of _defining_ the operations in this set of numbers. For this reason we have not stated postulates or definitions. Instead of posing the problem of how to _define_ multiplication of rational numbers, for instance, and then building toward this definition, we have proposed to the pupil that we examine the reasons why multiplication behaves as it *does*. We try to motivate the *usual rule* for multiplication, and then state this rule clearly and concisely. Instead of treating the commutative, associative, and distributive properties as postulates we simply point them out and state them clearly early in the discussion of the operations and then use them in subsequent discussion. It is hoped that as he follows this treatment the pupil will begin to see the skeleton of a mature mathematical approach to number systems without being smothered by the more subtle ideas which are necessary for complete understanding. Do *not* tell your pupils, however, that they are really just talking in a very naive way about something that is really somewhat sophisticated.

The student will probably bring a wealth of terminology and calculating procedures with him from the sixth grade. He is probably familiar with the terms mixed number, proper fraction, improper fraction, decimal fraction, and lowest terms. We suggest that you use the terms that are familiar to you and to your students. We have made no attempt to build such a vocabulary in this chapter because these terms are not used extensively in subsequent mathematics.

We wish to discourage the indiscriminate use of the operations of "cancelling" and "reducing fractions". Instead we stress the operations of "factoring" and "finding the simplest form for a fraction". Thus instead of

$$\frac{12}{15} \div \frac{3}{5} = \frac{\overset{4}{\cancel{12}}}{\underset{\cancel{5}}{\cancel{15}}} \cdot \frac{\cancel{5}}{3} = \frac{4}{3}$$

we prefer

$$\frac{12}{15} \div \frac{3}{5} = \frac{12}{15} \cdot \frac{5}{3} = \frac{4 \cdot 3}{5 \cdot 3} \cdot \frac{5}{3} = \frac{4 \cdot 3 \cdot 5}{5 \cdot 3 \cdot 3} = \frac{3}{3} \cdot \frac{5}{5} \cdot \frac{4}{3} =$$

$$1 \cdot 1 \cdot \frac{4}{3} = \frac{4}{3}.$$

The term "factor" has been used in an earlier chapter. Moreover in ninth grade algebra a student is asked to factor $x^2 - 1$.

We use freely the terminology and concepts developed in earlier chapters. Thus the terms factor, prime, greatest common factor, least common multiple, ray, segment and division algorithm are used.

In order to discuss the arithmetic operations intelligently, indeed to tell what the elements of our system of rational numbers are, we must first discuss the meaning of a fraction and the equality of fractions. We must point out that a fraction is a name for a number and that a number has many names. We shall not belabor the difference between a number and its name.

Various means are used to describe a rational number--measures of quantities, points on the number line, and quotients of whole numbers. We hope that you will find that we have introduced the number sentence $b \cdot x = a$ in an easy way. We realize that students will vary greatly in the amount of abstract mathematics they will comprehend. These different approaches can be viewed as three distinct explanations, or each can be viewed as an interpretation of the other. This chapter can serve as a bridge from arithmetic to algebra. Our concurrent use of the number line and of equations or number sentences is intermediate between a concrete interpretation of number and the abstract manipulation of number as represented by a symbol in an equation. You may want to point out that while piles of sand or points on the line are not rational numbers, problems involving them provide motivation for the abstract mathematical definitions which create the system of rational numbers. You may also want to give more "social-type" exercises and examples.

All pupils should be able to see that $a/b$ represents the number $x$ such that $b \cdot x = a$ and to know that a rational number is one which can be written as a quotient of a whole number and a counting number.

In connection with our use of the number sentence, you will note that the most elaborate equation discussed is of the form $ax = b$ or $a + x = b$. It is always the case that $x$ is to stand for a number. We are usually seeking a replacement for it, either from the set of whole numbers, or from the set of rational numbers. The symbol "x" is not to be regarded as a variable. In any number sentence or equation, the point is made that we have a representation of the same number in two different ways. When an operation is performed on this number, we get the same result, no matter which representation is used. Thus when an operation is performed on both sides of an equality, the equality remains true, although the numbers involved may be changed.

You may have other highly successful teaching methods for presenting the arithmetic operations with rational numbers. You may even have found that equations cannot be introduced in the seventh grade. We hope you will try to understand the approach we are making here. By simply regarding $a/b$ as the replacement for $x$ which makes $bx = a$ true, we think we are able to present sound mathematical reasons behind the arithmetic operations. We hope that you will find it a successful treatment in your classes. If it is successful, we think the pupil will find his subsequent mathematics easier and more meaningful because a more abstract point of view has been taken here.

## 6-1. Introduction

The first section introduces the idea of a mathematical system. In brief, this is a set of elements on which operations have been defined together with a list of the laws which these operations obey. We stress that the whole numbers are a system and that the rational numbers are a system and we point out the differences between these systems. Closure of the set of non-zero rational numbers under the division operation is the key property which motivates the construction of the rationals. In more advanced courses the rational numbers are constructed from the whole numbers and we are trying to convey the flavor of this technique, without being completely rigorous.

The historical remarks show that arithmetic concepts were introduced with some difficulty into ancient civilizations. Egyptian and Babylonian fractions will provide an opportunity for a brief review of the operations we think the students should already be able to perform.

## 6-2. The Rational Numbers and Their Names

Much of the material in this section is a review of ideas which were introduced in Chapter 3. The pupil should learn two basic ways of thinking of rational numbers: as measures of quantity in physical or social situations, and as quotients of whole numbers by counting numbers. Point out why the distinction is made between whole and counting number here. A whole number might be zero, and division by zero is not possible. Emphasize that the rational number a/b is the number whose product with b is a , and that a concise way of saying just this is: a/b is the number x for which $b \cdot x = a$.

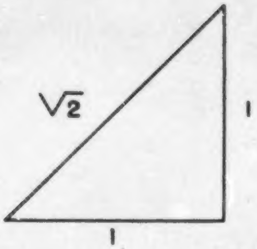
## 6-3. The Number Line

The discussion of the number line should convince the pupil that for every rational number there is a point on the number line. He should be given plenty of opportunity to practice locating rational numbers on the number line and finding the number which corresponds to a given point. The number line is an important intuitive device in the discussion of properties of numbers, and it has considerable other importance in its own right. Understanding of the number-point correspondence is indispensible in the study of measurement, graphs, and metric geometry later in this book. You may want to point out to your pupils that just as the edge of a ruler represents a line (as in Chapter 4), the edge and the numbers and marks on the ruler represent the number line. If a pupil should ask whether all points on the number line correspond to rational numbers, it should be easy to convince him that there is no rational number whose square is 2. The conventional proof goes as follows:

Suppose $\sqrt{2} = a/b$ where a and b are counting numbers. We may assume that a/b is in simplest form. It would then follow that $2 = \frac{a^2}{b^2}$, or that $2b^2 = a^2$. This is the statement that $a^2$ is an even number. From this it follows that a itself must be even, for if it were odd its square would be odd also, contrary to $a^2 = 2b^2$. Since a is even, we may write $a = 2c$. Hence $a^2 = 4c^2$. Replacing $a^2$ by $4c^2$ we could conclude that

$2b^2 = 4c^2$ and hence that $b^2 = 2c^2$. But then we could argue as before to show that $b^2$ is even and that b is also even. But now we have a contradiction: both a and b are even, whereas we initially assumed that a/b was in simplest form. So it is impossible that $\sqrt{2} = a/b$.

Later on, when they study the Pythagorean relation, they will learn that there is a point on the number line corresponding to



$\sqrt{2}$. $\sqrt{2}$ can be shown by marking off on the number line the hypotenuse of a right isosceles triangle whose legs are one unit long.

The number line will be used to develop the rules for arithmetical operations. To locate a rational number on the number line and to develop the meaning of mixed numbers use is made of the division algorithm involving whole numbers:

Dividend = (divisor · quotient) + remainder

where

$0 \leq$ remainder $<$ divisor.

It should be emphasized that this algorithm is really a familiar result about division which is succintly stated. Thus

$$\begin{array}{r} 6 \\ 7\,\overline{)\,43} \\ \underline{42} \\ 1 \end{array} \quad \text{is stated as: } 43 = (7 \cdot 6) + 1.$$

Attention should be called to the word "quotient". As used in the division algorithm it denotes the whole number part of the rational number that is obtained by division. It gives the largest whole number less than or equal to the rational number itself. Thus 43/7 is a rational number and is the quotient of 43 divided by 7.

$$43 = (7 \cdot 6) + 1 \qquad \text{and} \qquad 6 < 43/7 < 7$$

This ambiguity in the word "quotient" is regrettable, but it must be faced for it is regularly used in these two senses. The real point is that when we operate within the system of whole numbers, "quotient" means one thing, but when we operate within the system of rational numbers, it means another. The word "quotient" is ambiguous until the system within which it is used is specified.

## 6-4. Equality of Rational Numbers

In this section you might point out to your pupils the parallel between the relationship of numeral to number and the relationship of fraction to rational number. We will not insist on careful use of this distinction, however. The main result of this section is the property:

$$\frac{a}{b} = \frac{c}{d} \quad \text{if and only if} \quad a \cdot d = b \cdot c$$

We state, by example and by symbols, as a general rule, that

$$\frac{a}{b} = \frac{a \cdot k}{b \cdot k}.$$

We don't urge the pupils first to multiply $a/b$ by 1 in the form $k/k$. Logically we cannot suggest this now since multiplication of rationals has not been discussed. While this device does entail an additional step, after you discuss multiplication, it can be used to provide a review of both multiplication and equality.

Incidentally note that the phrase "in general" which is used in this chapter has a much stronger meaning than it does in everyday language. Contrast the remarks, "In general, I get up each morning at seven o'clock" and "In general, $a \cdot b = b \cdot a$".

## 6-5. Ordering of the Rational Numbers

The section on ordering is a natural successor to the section on equality because the criteria for "equality" and "less than" are so similar. It permits us to use the number line and should crystallize the concept of equality. After the arithmetical operations have been discussed, you may want to bring out which operations on inequalities preserve the sense of the inequality. Because of the length of this chapter we have omitted them and included some of this information in the exercises. The pertinent results are these:

If $r$ and $s$ are rational numbers such that $r < s$, then

(1) $(r + t) < (s + t)$, for all rational numbers $t$.

(2) $(r \cdot t) < (s \cdot t)$, for all (positive) rational numbers $t$.

(3) $(r - t) < (s - t)$, for all rational numbers $t$ (such that $t < r$)

*(4) $(t - s) < (t - r)$, for all rational numbers $t$ (such that $t > s$)

(5) $(r \div t) < (s \div t)$, for all (positive) rational numbers $t$ such that $t \neq 0$.

*(6) $(t \div s) < (t \div r)$, for all (positive) rational numbers $t$, provided $r \neq 0$.

The (*) denotes those operations in which the inequality is reversed.

The second part of section 5 is starred and can be omitted without loss of continuity. However, we feel that it will be interesting to many seventh graders. The pupils should be urged to make mental comparisons between rational numbers in cases where this can be done. It should be pointed out, however, that this cannot always be done. Sometimes it is necessary to use Property 3. The density property should be interesting to the pupils. It is important, too, for it is one of the few structural differences between the rational number system and the system of whole numbers which is encountered in this chapter.

## 6-6. The Multiplication Operation

The arithmetic operations will now be discussed. It is important to discuss the reasons for the rules we use for the operations. This is done here with great care. We urge you to stress the so-called algebraic treatment of these operations whenever you think it feasible for your classes. The treatment of multiplication from this point of view is confined to the last of the three parts in this section. It is hoped that the pupil will be familiar enough with the multiplication operation from the number line point of view by the time he reaches this part so that he can make the transition to the somewhat more sophisticated point of view without difficulty. The ideas in this third part of the section are important as preparation for the type of mathematical thinking which will come later. Therefore this sub-section should be omitted only in case of extreme necessity.

If the question of multiplication of mixed numbers should come up here, have the pupils treat them by first converting to improper fractions. Multiplication of mixed numbers is discussed in section 8 when the distributive property can be used.

We should like to caution both teacher and pupil against the slavish use of formulas like $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$. Thus to obtain 1/3 of 3/4 it is easy to view 3/4 as $3 \cdot (1/4)$ so that 1/3 of 3/4 is 1/4. Try viewing this as a pie cut into fourths. You are asked for a third of 3/4 of this pie. On the other hand the formula gives $\frac{1}{3} \cdot \frac{3}{4} = \frac{3}{12}$ and an extra step is required to get

$\frac{3}{12} = \frac{3 \cdot 1}{3 \cdot 4} = \frac{1}{4}.$

You are urged to accept as a correct answer to a problem any form for the correct rational number unless the form required for the answer has been specified in advance. $\frac{3}{12}$ is every bit as good an answer as 1/4, unless it has been specified that a fraction in simplest form is sought.

Since much attention is given to the admissibility of different names for a number, you may wonder if, for example "$\frac{1}{3} \times \frac{3}{4}$" is not an acceptable answer for the problem: Find the product of 1/3 and 3/4. The operational symbols +, -, ×, ÷ are often used to indicate the desired operation. They are usually used in this book to deonte the rational number obtained when the two numbers are combined by the operation in question. It is in this last sense that "$\frac{1}{3} \times \frac{3}{4}$" is a name for a rational number and must be accepted as a correct answer if specific directions for the desired form of the answer were not given. With your brighter students you may have to word your problems carefully if you do not wish to accept "$\frac{1}{3} \times \frac{3}{4}$" as a correct answer!

## 6-7. The Division Operation

You might point out here the parallel between a collection or "family" of facts about division which the children may have learned in elementary school and our approach to division. Thus the following columns of facts display a transition from elementary arithmetic to the treatment of the operations we are presenting here:

$$\begin{array}{lll} 8 \div 2 = 4 & a \div b = x & \frac{a}{b} \div \frac{c}{d} = x \\ 4 \cdot 2 = 8 & b \cdot x = a & \\ 2 \cdot 4 = 8 & x \cdot b = a & \frac{c}{d} \cdot x = \frac{a}{b} \\ 8 \div 4 = 2 & & x \cdot \frac{c}{d} = \frac{a}{b} \end{array}$$

You may want to add more emphasis than you find in this section on the technique of carrying out a division problem by expressing the quotient as a complex fraction and then making the denominator 1. Or you may want to emphasize the method illustrated in Problem 5 of Exercises 6-7b of converting a complex fraction to a simple fraction by multiplying the two denominators by their least common multiple.

## 6-8. The Addition Operation

To point up the necessity for looking at the "why" of the arithmetic operations carefully we may pose, as a *definition* for addition, the more easily computed

$$\frac{a}{b} + \frac{c}{d} = \frac{a + c}{b + d} .$$

"Why don't we add this way? It would be easier!"

This definition gives an operation which is commutative, associative, and multiplication is distributive over this "addition".

However, while it is true that $\frac{(a + c)}{(b + d)}$ is a rational number, this "operation" has two defects which make it useless. If we were to use it, we would have, for example, $\frac{1}{2} + \frac{1}{3} = \frac{2}{5}$ while

$\frac{2}{4} + \frac{3}{9} = \frac{5}{13}$ yet $\frac{2}{5} \neq \frac{5}{13}$.

Thus this "addition" is not well defined, for the names we use for numbers affect their sum. It is also easy to show by example that this kind of addition does not correspond to our intuitive concept of adding quantities. It is important to discuss why the definitions are made as they are.

We can make no pretense of verifying all the details necessary for the development of the rational numbers from the counting numbers. You might stimulate a good class by showing that our rules for addition and multiplication are independent of the fractions used to represent the numbers involved.

Emphasize the fact that there are two ways in which pupils can carry out an addition or multiplication of mixed numbers and encourage them to try both methods in many examples. Point out to pupils how they are using the commutative, associative, and distributive properties whenever they use the method of treating the whole number and fraction part separately.

## 6-9. The Subtraction Operation

Here again it might be well to keep in mind the transition from the ideas which pupils have learned in arithmetic to the point of view which we are presenting here which is illustrated by the following columns of facts concerning subtraction:

$$8 - 2 = 6$$
$$6 + 2 = 8$$
$$2 + 6 = 8$$
$$8 - 6 = 2$$

$$b - a = x$$
$$a + x = b$$
$$x + a = b$$

$$\frac{c}{d} - \frac{a}{b} = x$$
$$\frac{a}{b} + x = \frac{c}{d}$$
$$x + \frac{a}{b} = \frac{c}{d}$$

Again, encourage the pupils to try subtracting mixed numbers both by converting them first to improper fractions with a common denominator and by working separately with the whole number and fraction parts.

## Answers to Exercises 6-1

1. (a) $\underline{3} \cdot x = \underline{6}$

   (b) $\underline{2} \cdot x = \underline{5}$

   (c) $\underline{3} \cdot x = \underline{9}$

   (d) $\underline{6} \cdot x = \underline{18}$

   (e) $\underline{4} \cdot x = \underline{10}$

   (f) $\underline{1} \cdot x = \underline{7}$

2. $x$ is a whole number in (a), (c), (d) and (f): In (a) $x = 2$, in (c) $x = 3$, in (d) $x = 3$, in (f) $x = 7$.

3. (a) $x = \frac{3}{2}$ (e) $x = 7$

   (b) $x = 8$ (f) $x = \frac{11}{5}$

   (c) $x = \frac{15}{4}$ (g) $x = 6$

   (d) $x = 8$ (h) $x = \frac{1}{8}$

4. (a) If $x$ is the number of cookies each boy should receive, then $3 \cdot x = 12$.

   (b) If $x$ is the number of miles he drove on each gallon of gasoline used, then $10 \cdot x = 160$

   (c) If $x$ is the number of bags of cement needed for each foot of walk, then $30 \cdot x = 20$.

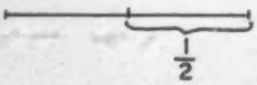
## Answers to Exercises 6-2a

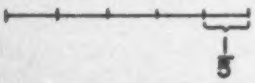
1. (a) 1, 2, 3, etc. (b) 0, 1, 2, 3, etc. (c) 0

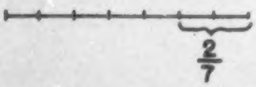
   (d) Any fraction in which the denominator is not a factor of the numerator.

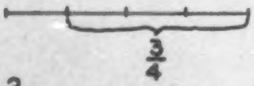
2. All except (g)

3. (a), (b), (e), (h)

4. (a) ; $2x = 1$

   (b) ; $5x = 1$

   (c) ; $7x = 2$

   (d) ; $4x = 3$

5. (a) $\frac{3}{2}$ may mean: (1) That 3 units were divided into 2 equal parts and one part was taken.

   (2) That each unit was divided into 2 equal parts and 3 of them were taken.

(3) The number x that makes the number sentence $2x = 3$ true.

(b) $\frac{2}{5}$ may mean: (1) That 2 units were divided into 5 equal parts and one part was taken.

(2) That each unit was divided into 5 equal parts and 2 of these parts were taken.

(3) The number that makes the number sentence $5x = 2$ true.

(c) $\frac{3}{3}$ may mean: (1) That 3 units were divided into 3 equal parts and one part was taken.

(2) That each unit was divided into 3 equal parts and 3 of these parts were taken.

(3) The number that makes the number sentence $3x = 3$ true.

(d) $\frac{0}{5}$ may mean: (1) That 0 units were divided into 5 equal parts and one part was taken.

(2) That each unit was divided into 5 equal parts and 0 of these parts were taken.

(3) The number that makes the number sentence $5x = 0$ true.

## Answers to Exercises 6-2b

1. (a) $\frac{1}{2}$ (b) $\frac{1}{8}$ (c) $\frac{1}{5}$ (d) $\frac{1}{51}$ (e) $x = 1$

(f) $x = 4$ (g) $x = \frac{1}{117}$ (h) $x = 13$

2. (a) $x = 9$ (c) $x = 19$ (e) $x = 0$ (g) $x = 6$

(b) $x = 15$ (d) $x = 2$ (f) $x = 3$ (h) $x = \frac{3}{2}$

3. (a) Any ten of the following:

7, 14, 21, 28, 35, 42, 49, 56, 63, 70, 77, etc.

(b) Any five of the following:

26, 39, 52, 65, 78, etc.

4. (a) 7 (c) 84 (e) $\frac{14}{2}$ or 7 (g) 84

(b) 8 (d) $\frac{12}{3}$ or 4 (f) $\frac{56}{7}$ or 8 (h) $\frac{12}{3}$ or 4

*5. a = e, b = f, c = g, d = f

The children might say something like the following:

(a) When you add fractions with the same denominator, you add the numerators and place the result over the common denominator.

(b) When you subtract fractions with the same denominator, you subtract the numerators and place the result over the common denominator.

(c) When you multiply fractions, you multiply the numerators and put the product over the product of the denominators.

(d) When you divide a fraction by a fraction you invert the divisor and multiply.

6. If n = 2 then d = 3 $\frac{n}{d} = \frac{2}{3}$

If n = 4 then d = 6 $\frac{n}{d} = \frac{4}{6} = \frac{2}{3}$

If n = 6 then d = 9 $\frac{n}{d} = \frac{6}{9} = \frac{2}{3}$

If n = 8 then d = 12 $\frac{n}{d} = \frac{8}{12} = \frac{2}{3}$

If n = 10 then d = 15 $\frac{n}{d} = \frac{10}{15} = \frac{2}{3}$

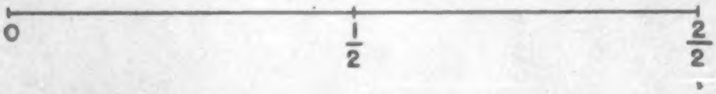
3n = 2d if and only if $\frac{n}{d} = \frac{2}{3}$

7.

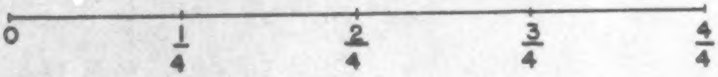
$$\frac{14}{5+\frac{10}{3+\frac{4}{2}}} = \frac{14}{5+\frac{10}{3+2}} = \frac{14}{5+\frac{10}{5}} = \frac{14}{5+2} = \frac{14}{7} = 2$$

## Answers to Exercises 6-3

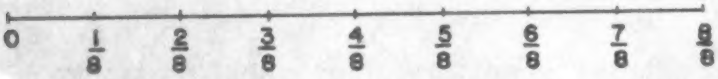
1. (a)



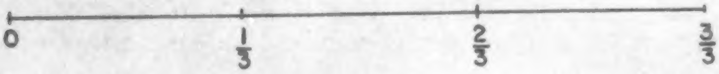
(b)



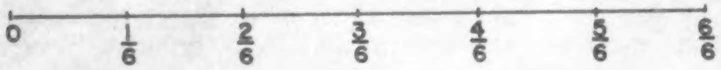
(c)


0
1/8
2/8
3/8
4/8
5/8
6/8
7/8
8/8

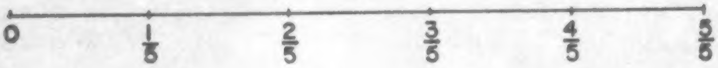
(d)



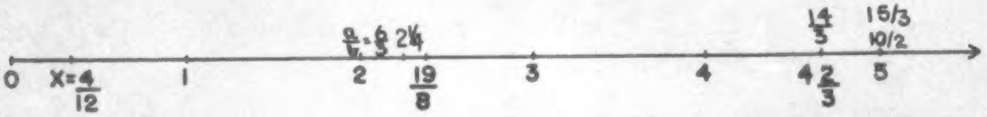
(e)


0
1/6
2/6
3/6
4/6
5/6
6/6

(f)


0
1/5
2/5
3/5
4/5
5/5

2.


0
X=4/12
1
2
8/4 = 6/3
2¼
19/8
3
4
14/3
4 2/3
15/3
10/2
5

3. (a)

A: $\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$

B: $\frac{1}{4} = \frac{2}{8} = \frac{3}{12}$

C: $\frac{3}{4} = \frac{6}{8} = \frac{9}{12}$

D: $\frac{1}{3} = \frac{2}{6} = \frac{3}{9}$

E: $\frac{2}{3} = \frac{4}{6} = \frac{6}{9}$

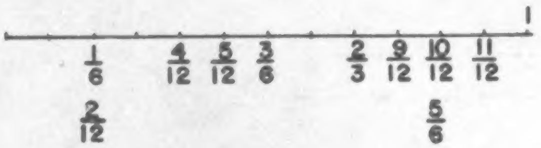
F: $\frac{1}{1} = \frac{2}{2} = \frac{3}{3}$

(b) The rational number located at point B is less than the rational number located at point A. Because point B is to the left of point A on the number line.

(c) Point A is to the left of point C. Therefore the rational number located at C is greater than the one located at A.

## Answers to Exercises 6-3b

1.


1
1/6
2/12
4/12
5/12
3/6
2/3
9/12
10/12
5/6
11/12

2. (a) $\frac{2}{3} < \frac{9}{12}$ (d) $\frac{3}{6} > \frac{4}{12}$

(b) $\frac{1}{6} = \frac{2}{12}$ (e) $\frac{11}{12} > \frac{5}{6}$

(c) $\frac{2}{3} < \frac{10}{12}$ (f) $\frac{3}{6} > \frac{5}{12}$

3. (a) 5, 12 (b) $\frac{7}{4}$ (c) 6 (d) 5, 3, 4 (e) 2·8, 6·7

(f) $\frac{6 + 8}{2}$ or 7 (g) $\frac{18}{12}$ or $\frac{3}{2}$ (h) $\frac{9 + 12}{3}$ or 7 (i) $\frac{2}{3}$

## Answers to Class Discussion Exercises 6-4

1. (a) $\frac{2}{3}$ (c) $\frac{6}{9}$

(b) $\frac{4}{6}$ (d) $\frac{10}{15}$

2. If the numerator and denominator of any fraction are multiplied by the same counting number, the number represented by the fraction is not changed.

## Answers to Exercises 6-4a

1. (a) G.C.F. is 2; $\frac{8}{10} = \frac{4}{5}$ (g) G.C.F. is 1; $\frac{55}{21} = \frac{55}{21}$

(b) G.C.F. is 3; $\frac{9}{12} = \frac{3}{4}$ (h) G.C.F. is 3; $\frac{9}{6} = \frac{3}{2}$

(c) G.C.F. is 3; $\frac{21}{24} = \frac{7}{8}$ (i) G.C.F. is 3·7; $\frac{105}{147} = \frac{5}{7}$

(d) G.C.F. is 9·4; $\frac{36}{72} = \frac{1}{2}$ (j) G.C.F. is 3; $\frac{165}{117} = \frac{55}{39}$

(e) G.C.F. is 3·5; $\frac{15}{510} = \frac{1}{34}$ *(k) G.C.F. is 5·11; $\frac{10395}{17875} = \frac{189}{325}$

(f) G.C.F. is 13; $\frac{13}{78} = \frac{1}{6}$

2. $\frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \frac{8}{12} = \frac{10}{15}$ etc.

3. $\frac{c}{d} = \frac{a \cdot k}{b \cdot k}$ where k is any counting number.

## Answers to Exercises 6-4b

1. (a) $\frac{6}{21} = \frac{10}{35} = \frac{2}{7}$, $6 \cdot 35 = 10 \cdot 21 = 210$

   (b) $\frac{3}{2} = \frac{45}{30}$, $3 \cdot 30 = 2 \cdot 45 = 90$

   (c) $\frac{9}{15} = \frac{15}{25} = \frac{3}{5}$, $9 \cdot 25 = 15 \cdot 15 = 225$

   (d) $\frac{22}{18} = \frac{55}{45} = \frac{11}{9}$, $22 \cdot 45 = 18 \cdot 55 = 990$

2. (a) Does $13 \cdot 209 = 19 \cdot 143$?
   Yes. $2717 = 2717$

   (b) Does $13 \cdot 31 = 7 \cdot 59$?
   No. $403 \neq 413$

   (c) Does $55 \cdot 273 = 39 \cdot 385$?
   Yes. $15015 = 15015$

   (d) Does $11 \cdot 89 = 19 \cdot 51$?
   No. $979 \neq 969$

3. (a) False. $\frac{2}{4} = \frac{3}{6}$; both fractions represent the same rational number but they are different names.

   (b) False. $\frac{1}{3} \neq \frac{1}{4}$; but both fractions have equal numerators.

   (c) True. $\frac{2}{3} = \frac{4}{6}$; the fractions represent the same rational rational number and yet $2 \neq 4$.

   (d) False. Let $a = 1$, $b = 2$, $k = 1$.

   Then $\frac{a}{b} = \frac{1}{2}$

   $$\frac{a + k}{b + k} = \frac{1 + 1}{2 + 1} = \frac{2}{3}$$

   and $\frac{1}{2} \neq \frac{2}{3}$

4. (a) Yes. Let $k = 0$

Then $\frac{a}{b} = \frac{a+k}{b+k}$ since $\frac{a+0}{b+0} = \frac{a}{b}$

*(b) If $\frac{a}{b} = \frac{a+1}{b+1}$, then a must equal b.

e.g. let $a = b = 2$.

Then $\frac{2}{2} = \frac{2+1}{2+1} = \frac{3}{3}$

5. (a) $x = 6$ (e) $x = 10$ (h) $x = 28$

(b) $x = 28$ (f) $x = 2$ (i) $x = 90$

(c) $x = 10$ (g) $x = 7$ (j) $x = 10$

(d) $x = 4$ *(k) $x = \frac{ac}{b}$

6. (a) 100 (c) $60^2$ (e) 1296

(b) 84 (d) $60^3$ (f) 154

7. (a) $\frac{15}{100}, \frac{28}{100}$ (c) $\frac{360}{3600}, \frac{40}{3600}$

(b) $\frac{63}{84}, \frac{60}{84}$ (d) $\frac{297}{154}, \frac{217}{154}$

8. (a)

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\varphi(n)$ | 1 | 1 | 2 | 2 | 4 | 2 | 6 | 4 | 6 | 4 | 10 | 4 | 12 | 6 | 8 | 8 | 16 | 6 | 18 | 8 |

(b) $\varphi(3) \cdot \varphi(4) = \varphi(12)$

$(2) \cdot (2) = 4$

$\varphi(2) \cdot \varphi(5) = \varphi(10)$

$(1) \cdot (4) = 4$

$\varphi(a) \cdot \varphi(b) = \varphi(ab)$ if and only if the G.C.F. of a and b is 1. Also $\varphi(p) = (p - 1)$ if and only if p is prime.

(c) $$\varphi(1) + \varphi(2) + \varphi(3) + \varphi(6) = 6$$
$$1 + 1 + 2 + 2 = 6$$

$$\varphi(1) + \varphi(2) + \varphi(5) + \varphi(10) = 10$$
$$1 + 1 + 4 + 4 = 10$$

$$\varphi(1) + \varphi(3) + \varphi(5) + \varphi(15) = 15$$
$$1 + 2 + 4 + 8 = 15$$

$$\varphi(1) + \varphi(2) + \varphi(4) + \varphi(8) = 8$$
$$1 + 1 + 2 + 4 = 8$$

$$\because \ \varphi(1) + \varphi(F_1) + \varphi(F_2) + \varphi(F_3) + \dots + \varphi(n) = n$$

where the $F_1$ are all the possible factors of $n$ exclusive of 1 and $n$.

## Answers to Exercises 6-5a

1. (a) $\frac{26}{41} < \frac{7}{11}$; $\frac{7}{11} = \frac{287}{451}$, $\frac{26}{41} = \frac{286}{451}$

   (b) $\frac{27}{17} < \frac{8}{5}$; $\frac{8}{5} = \frac{136}{85}$, $\frac{27}{17} = \frac{135}{85}$

   (c) $\frac{9}{14} < \frac{47}{73}$; $\frac{9}{14} = \frac{657}{1022}$, $\frac{47}{73} = \frac{658}{1022}$

   (d) $\frac{16}{13} < \frac{37}{30}$; $\frac{16}{13} = \frac{480}{390}$, $\frac{37}{30} = \frac{481}{390}$

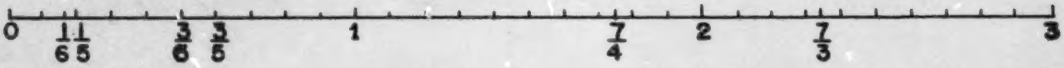
2. (a) $\frac{7}{5} < \frac{41}{29} < \frac{99}{70} < \frac{17}{12}$

   (b) $\frac{69}{22} < \frac{91}{29} < \frac{135}{43} < \frac{22}{7}$

3. (a) $\frac{4}{10}$ (b) 4

*4. John should take $\frac{7}{11}$ of the apple pie or 2/7 of the lemon pie and $\frac{4}{11}$ of the apple pie.

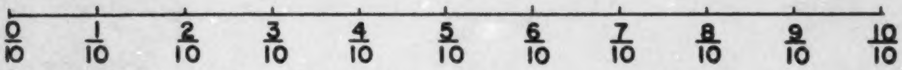
5. (a) $\frac{1}{5} > \frac{1}{6}$, $\frac{3}{5} > \frac{3}{6}$, $\frac{7}{4} < \frac{7}{3}$, $\frac{181}{300} > \frac{181}{301}$,



(b) The fraction with the smaller denominator will represent the larger number. For example, 7/4 can be thought of on the number line as follows: divide 7 into 4 equal parts and take one of them. Similarly, 7/5 would be represented by one of the pieces when 7 was divided into 5 equal parts. When there are more parts, each part must be smaller,

thus $\frac{7}{5} < \frac{7}{4}$.

6.



If we divide a unit segment into 10 equal parts or slots, we have the diagram above. If 11 points are chosen on the unit segment, at least 2 of the points must be in the same slot and the numbers they represent will differ by less than 1/10.

## Answers to Exercises 6-5b

1. (a) $\frac{3}{8} < \frac{4}{7}$, since $\frac{3}{8} < \frac{4}{8} < \frac{4}{7}$

(b) $\frac{5}{8} < \frac{76}{119}$, since $\frac{5}{8} = \frac{75}{120} < \frac{76}{120} < \frac{76}{119}$

(c) $\frac{120}{230} > \frac{116}{237}$; since $\frac{120}{230} > \frac{116}{230} > \frac{116}{237}$

(d) $\frac{119}{241} < \frac{1}{2}$, since $\frac{119}{241} < \frac{120}{241} < \frac{120}{240} = \frac{1}{2}$

*(e) $\frac{a}{b} < \frac{a+1}{b-2}$, since $\frac{a}{b} < \frac{a+1}{b} < \frac{a+1}{b-2}$

2. $\frac{7}{10} < \frac{8}{10}$, $\frac{7}{10} > \frac{1}{10}$, $\frac{7}{10} > \frac{2}{5}$, $\frac{7}{10} > \frac{1}{2}$, $\frac{7}{10} < \frac{3}{4}$, $\frac{7}{10} > \frac{3}{7}$,

$\frac{7}{10} > 0$, $\frac{7}{10} = \frac{21}{30}$, $\frac{7}{10} < \frac{7}{5}$, $\frac{7}{10} < \frac{7}{6}$, $\frac{7}{10} < \frac{7}{8}$, $\frac{7}{10} > \frac{7}{11}$, $\frac{7}{10} > \frac{7}{12}$

3. (a) 1/3 (b) 1/4 (c) 1/3 (d) 1/3

4. For each of these parts there are many correct answers. One of the simpler correct answers is given below in each case.

(a) $\frac{4}{10}$
(b) $\frac{4}{5}$
(c) $\frac{9}{24}$
(d) $\frac{9}{24}$
(e) $\frac{4}{5}$
(f) $\frac{7}{20}$
(g) $\frac{15}{28}$
(h) $\frac{17}{48}$
(i) $\frac{9}{10}$
(j) $\frac{251}{60}$

5. $\frac{2a + 1}{2b}$

*6. Since $\frac{a}{b}$ and $\frac{c}{d}$ are different numbers, one of them must be larger. Suppose it is $\frac{a}{b}$. Then $\frac{c}{d} < \frac{a}{b}$ or $bc < ad$. Then $ad - bc$ is at least 1, and $2ad - 2bc$ is at least 2. But then

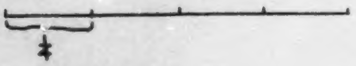
$$\frac{c}{d} = \frac{2bc}{2bd} < \frac{2bc + 1}{2\,bd} < \frac{2ad}{2bd} = \frac{a}{b}$$

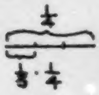
so that $\frac{2bc + 1}{2bd}$ lies between $\frac{c}{d}$ and $\frac{a}{b}$.

## Answers to Exercises 6-6a

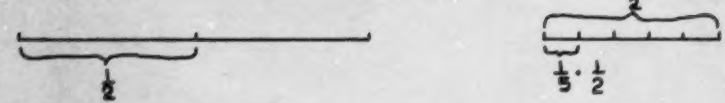
1. (a) $\frac{1}{12}$ (b) $\frac{1}{10}$ (c) $\frac{1}{10}$ (d) $\frac{1}{24}$ (e) $\frac{3}{5}$ (f) 1

2. (a)

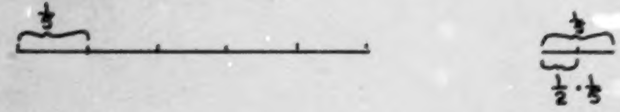




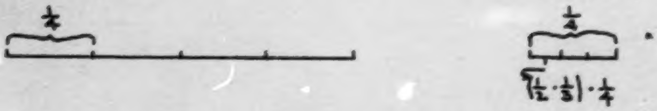
(b)



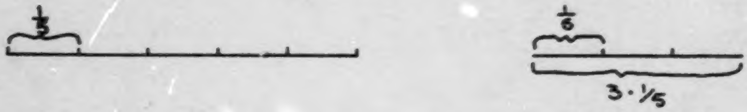
(c)



(d)



*(e)



## Answers to Class Discussion Problems 6-6

15; $\frac{1}{15}$ or $\frac{1}{3} \cdot \frac{1}{5}$; $\frac{1}{15}, \frac{2}{15}, \frac{3}{15}, \frac{12}{15}$; 12; 4; 8

## Answers to Exercises 6-6b

1. (a) $\frac{1}{6}$ (b) $\frac{2}{15}$ (c) $\frac{3}{20}$ (d) $\frac{6}{25}$ (e) $\frac{3}{100}$

(f) $\frac{12}{1000}$ or $\frac{3}{250}$ or $\frac{6}{500}$ (g) $\frac{9}{9}$ or 1 (h) $\frac{4}{9}$

(i) $\frac{4}{9}$ (j) $\frac{4}{3}$ or $1\frac{1}{3}$ (k) $\frac{1}{9}$ (l) $\frac{12}{2}$ or 6

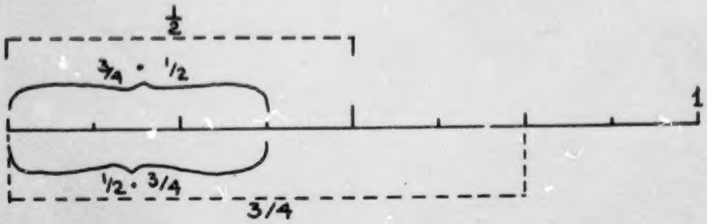
(m) $\frac{24}{4}$ or 6 (n) $\frac{1}{24}$ (o) $\frac{1}{1000}$ (p) $\frac{ac}{b}$

(q) $\frac{1}{a^2}$ (r) $\frac{1}{10,000}$ (s) $\frac{1}{1000}$ (t) 10 or $\frac{100}{10}$

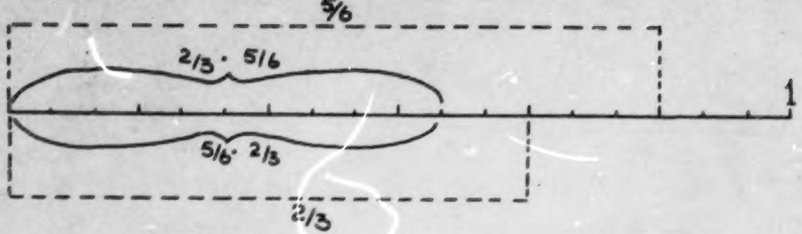
(u) 20 or $\frac{40}{2}$ (v) $\frac{1}{6}$ or $\frac{15}{90}$ or $\frac{5}{30}$ (w) 1 or $\frac{6}{6}$ or $\frac{30}{30}$

(x) $5\frac{1}{4}$ or $\frac{21}{4}$ (y) $\frac{1}{bc}$ (z) $\frac{ac}{bd}$

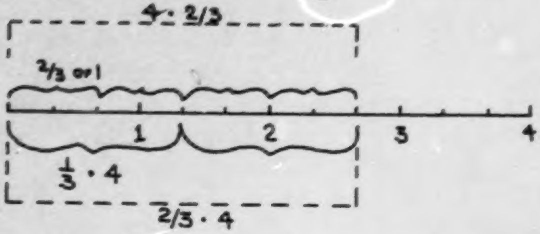
2. (a)



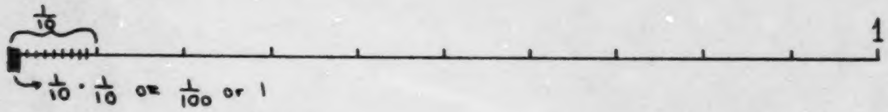
(b)


5/6
2/3 · 5/6
1
5/6 · 2/3
2/3

(c)


4 · 2/3
2/3 or 1
1
2
3
4
1/3 · 4
2/3 · 4

(d)


1/10
1
1/10 · 1/10 or 1/100 or 1

(e) Commutative property

3. (a) $\frac{1}{39}$

(b) $\frac{1}{2^3} \cdot \frac{1}{3 \cdot 5} = \frac{1}{2^3 \cdot 3 \cdot 5} = \frac{1}{120}$

(c) $\frac{1}{2^2 \cdot 3} \cdot \frac{1}{5} = \frac{1}{2^2 \cdot 3 \cdot 5} = \frac{1}{60}$

(d) $\frac{1}{5} \cdot 2 \cdot 5^2 = \frac{5}{5} \cdot 2 \cdot 5 = 1 \cdot 2 \cdot 5 = 10$

(e) $\frac{2}{3} \cdot \frac{1}{2^3} = \frac{2}{2} \cdot \frac{1}{2^2 \cdot 3} = 1 \cdot \frac{1}{2^2 \cdot 3} = \frac{1}{12}$

(f) $\frac{5}{2 \cdot 3} \cdot 2^3 \cdot 3 = \frac{2}{2} \cdot \frac{3}{3} \cdot 5 \cdot 2^2 = 1 \cdot 1 \cdot 5 \cdot 2^2 = 20$

(g) $2^2 \cdot 3^2 \cdot \frac{1}{2^2 \cdot 3} = \frac{2^2}{2^2} \cdot \frac{3}{3} \cdot 3 = 1 \cdot 1 \cdot 3 = 3$

(h) $\frac{2^2}{7} \cdot \frac{1}{2^2 \cdot 3} = \frac{2^2}{2^2} \cdot \frac{1}{7 \cdot 3} = 1 \cdot \frac{1}{7 \cdot 3} = \frac{1}{21}$

(i) $2^3 \cdot 3^2 \cdot \frac{5}{3^2} = \frac{3^2}{3^2} \cdot 2^3 . 5 = 1 \cdot 2^3 \cdot 5 = 40$

(j) $\frac{2^3}{3^2} \cdot \frac{17}{17} = \frac{2^3}{3^2} \cdot 1 = \frac{8}{9}$

(k) $\frac{2^3}{3^2} \cdot 1 = \frac{8}{9}$

(l) $\frac{3^2}{2^4} \cdot \frac{7}{2 \cdot 5} = \frac{3^2 \cdot 7}{2^4 \cdot 2 \cdot 5} = \frac{3^2 \cdot 7}{2^5 \cdot 5} = \frac{63}{160}$

(m) $\frac{13}{3 \cdot 5} \cdot \frac{2 \cdot 5}{3 \cdot 13} = \frac{13}{13} \cdot \frac{5}{5} \cdot \frac{2}{3^2} = \frac{2}{9}$

(n) $\frac{3 \cdot 11}{2^2 \cdot 5^2} \cdot \frac{2 \cdot 3^2 \cdot 5}{3^2 \cdot 11} = \frac{2}{2} \cdot \frac{3^2}{3^2} \cdot \frac{5}{5} \cdot \frac{11}{11} \cdot \frac{3}{2 \cdot 5} = 1 \cdot 1 \cdot 1 \cdot 1 \cdot \frac{3}{10} = \frac{3}{10}$

(o) $\frac{17}{3} \cdot \frac{3}{3 \cdot 17} = \frac{3}{3} \cdot \frac{17}{17} \cdot \frac{1}{3} = 1 \cdot 1 \cdot \frac{1}{3} = \frac{1}{3}$

(p) $\left(\frac{3^2}{2^3} \cdot \frac{2^3}{3}\right) \cdot \frac{2^2}{5} = \left(\frac{2^3}{2^3} \cdot \frac{3}{3} \cdot \frac{3}{1}\right) \cdot \frac{2^2}{5} = (1 \cdot 1 \cdot 3) \cdot \frac{4}{5} =$

$3 \cdot \frac{4}{5} = \frac{12}{5} = 2\frac{2}{5}$

(q) $\frac{3^2}{2^3} \cdot \left(\frac{2^3}{3} \cdot \frac{2^2}{5}\right) = \frac{3^2}{2^3} \cdot \frac{2^5}{3 \cdot 5} = \frac{2^3}{2^3} \cdot \frac{3}{3} \cdot \frac{3 \cdot 2^2}{5} = \frac{12}{5} = 2\frac{2}{5}$

(r) The answers are the same.

(s) $\frac{7}{2 \cdot 5} \cdot \frac{2^2 \cdot 5}{3 \cdot 7} \cdot 2^2 \cdot 5^2 = \frac{2}{2} \cdot \frac{5}{5} \cdot \frac{7}{7} \cdot \frac{2^3 . 5^2}{3} = 1 \cdot 1 \cdot 1 . \frac{200}{3} = 66\frac{2}{3}$

(t) $\frac{2}{3} \cdot 2^2 \cdot 3 \cdot 5 \cdot \frac{5}{2} = \frac{2}{2} \cdot \frac{3}{3} \cdot \frac{2^2 \cdot 5^2}{1} = 1 \cdot 1 \cdot \frac{100}{1} = 100$

4. d through i

5. (a) $1\frac{7}{8}$ (b) $21\frac{1}{3}$ (c) $8\frac{8}{9}$ (d) 70 (e) $8\frac{1}{4}$

(f) $6\frac{7}{8}$ (g) $5\frac{1}{4}$ (h) $1\frac{1}{2}$ (i) 39 (j) $60\frac{5}{7}$

## Answers to Exercises 6-6c

1. (a) 3, 2
5, 1

(b) $x \cdot y = \frac{2}{3} \cdot \frac{1}{5}$ ;

$$\begin{aligned} 3x \cdot 5y &= 2 \cdot 1 \\ 3 \cdot 5 \cdot x \cdot y &= 2 \cdot 1 \\ x \cdot y &= \frac{2 \cdot 1}{3 \cdot 5} \end{aligned}$$

Therefore $\frac{2}{3} \cdot \frac{1}{5} = \frac{2 \cdot 1}{3 \cdot 5}$

2. (a) $x = 5$

(b) $x = 8$

3. $r = \frac{3}{4}$

4. $\frac{a}{b} \cdot \frac{k}{k} = \frac{ak}{bk}$ The number 1 can be expressed as $\frac{k}{k}$.

*5. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} = \frac{ca}{db} = \frac{c}{d} \cdot \frac{a}{b}$

*6. $\left(\frac{a}{b} \cdot \frac{c}{d}\right) \cdot \frac{e}{f} = \frac{ac}{bd} \cdot \frac{e}{f} = \frac{ace}{bdf} = \frac{a}{b} \cdot \frac{ce}{df} = \frac{a}{b}\left(\frac{c}{d} \cdot \frac{e}{f}\right)$

7. True; yes.

8. $\frac{1}{4}$ pint left

9. (a) $\frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$ cup orange juice

$\frac{2}{3} \cdot \frac{1}{4} = \frac{1}{6}$ cup lemon juice

$\frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$ cup grapefruit juice

$\frac{2}{3} \cdot 1 = \frac{2}{3}$ cup pineapple juice

$\frac{2}{3}\cdot\frac{3}{2} = 1$ cup water

$\frac{2}{3}\cdot\frac{1}{2} = \frac{1}{3}$ cup syrup

(b) 8 persons would require $\frac{8}{6}$ or $\frac{4}{3}$ or $1\frac{1}{3}$ times the amounts in the original recipe, or 2 times the amount of the recipe for 4 persons.

$\frac{4}{3}\cdot\frac{1}{2} = \frac{2}{3}$ cup orange juice

$\frac{4}{3}\cdot\frac{1}{4} = \frac{1}{3}$ cup lemon juice

$\frac{4}{3}\cdot\frac{1}{2} = \frac{2}{3}$ cup grapefruit juice

$\frac{4}{3}\cdot 1 = 1\frac{1}{3}$ cup pineapple juice

$\frac{4}{3}\cdot\frac{3}{2} = 2$ cup water

$\frac{4}{3}\cdot\frac{1}{2} = \frac{2}{3}$ cup syrup

10. $1\frac{5}{8}\cdot 10 = 16\frac{1}{\text{[illegible]}}$ miles

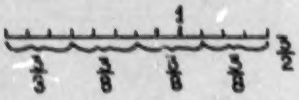
## Answers to Exercises 6-7a

1. (a) $14x = 52$ (c) $4x = \frac{2}{3}$ (e) $\frac{21}{115}x = 0$
(b) $\frac{1}{2}x = \frac{3}{8}$ (d) $\frac{4}{5}x = 18$ (f) $\frac{c}{d}x = \frac{a}{b}$

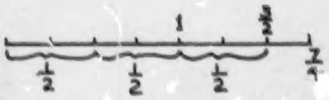
2. (a) $x = \frac{4}{5}$ (c) $x = 8 \div \frac{3}{5}$
(b) $x = \frac{5}{9} \div \frac{2}{3}$ (d) $x = \frac{5}{3} \div 6$

3. (a) $\frac{16}{15}$ (c) $9$ (e) $\frac{1}{(3+4)(5+2)}$
(b) $\frac{6}{5}$ (d) $\frac{1}{12}$ (f) $(a + b)$

4. (a) (b)



$\frac{3}{2} \div \frac{3}{8} = 4$



$\frac{7}{4} \div \frac{1}{2} = 3\frac{1}{2}$

## Answers to Exercises 6-7b

1. (a) 1 (b) $\frac{1}{9}$ (c) 16 (d) 72 (e) $\frac{1}{100}$

(f) 1 (g) $\frac{25}{81}$ (h) $\frac{1}{1000}$ (i) $\frac{1}{2}$ (j) $\frac{b}{a}$

(k) $\frac{ad}{bc}$ (l) $\frac{1}{12}$ (m) 1 (n) 4 (o) 90

(p) 100 (q) $\frac{7}{8}$

2. (a) $\frac{1}{10}$ (b) $\frac{3}{4}$ (c) $\frac{1}{72}$ (d) $\frac{9}{4}$ or $2\frac{1}{4}$ (e) $\frac{2}{17}$

(f) 1 (g) $\frac{ab}{b}$ or a (h) $\frac{ad}{bc}$ (i) $\frac{\frac{2}{3}}{\frac{4}{7}}$ or $\frac{14}{12}$ or $\frac{7}{6}$ or $1\frac{1}{6}$

3. (a) 2 or $\frac{8}{4}$ (b) 75 (c) 8 or $\frac{24}{3}$ (d) $\frac{1}{32}$

(e) $\frac{35}{48}$ (f) $\frac{3}{2}$ or $1\frac{1}{2}$ or $\frac{396}{264}$ (g) $\frac{1}{ac}$ (h) ca

4. (a) $\frac{1}{9}$ (b) $\frac{3}{11}$ (c) $\frac{1}{384}$ (d) 190 (e) 1

(f) $\frac{25}{11}$ or $2\frac{3}{11}$ (g) $\frac{54}{28}$ or $\frac{27}{14}$ or $1\frac{13}{14}$ (h) 10

5. (a) L.C.M. is 10; $\frac{3}{2}$

(b) L.C.M. is 16; $\frac{3}{4}$

(c) L.C.M. is 20; $\frac{24}{19}$ or $1\frac{5}{19}$

(d) L.C.M. is 100; $\frac{43}{70}$

6. (a) $\frac{7}{9}$ (b) $\frac{4}{7}$, The division operation does not have the associative property.

7. (a) $\frac{4}{3}$ (b) $\frac{3}{4}$, No. Division is not a commutative operation.

8. (a) 1 (b) 1

9. The product is 1. Therefore $\frac{a}{b} \div \frac{c}{d}$ and $\frac{c}{d} \div \frac{a}{b}$ are reciprocals of each other.

10. (a) 10 times as large.
    (b) $\frac{16}{49}$ as large.

## Answers to Exercises 6-8a

1. (a) $\frac{6}{3}$ or 2 (d) $4\frac{3}{8}$ or $\frac{35}{8}$ (g) $\frac{17}{16}$ or $1\frac{1}{16}$
   (b) $\frac{6}{3}$ or 2 (e) $\frac{7}{10}$ (h) $\frac{17}{16}$ or $1\frac{1}{16}$
   (c) $4\frac{3}{8}$ or $\frac{35}{8}$ (f) $\frac{7}{10}$

2. Yes.

3. (a) $\frac{7}{3}$ or $2\frac{1}{3}$ (d) $\frac{15}{8}$ or $1\frac{7}{8}$ (g) $\frac{52}{15}$ or $3\frac{7}{15}$
   (b) $\frac{7}{3}$ or $2\frac{1}{3}$ (e) $3\frac{11}{30}$ or $\frac{101}{30}$ (h) $\frac{52}{15}$ or $3\frac{7}{15}$
   (c) $\frac{15}{8}$ or $1\frac{7}{8}$ (f) $\frac{101}{30}$ or $3\frac{11}{30}$

4. Yes.

*5. (a) Yes. (b) The commutative property.

*6. (a) Yes. (b) The associative property.

7. (a) $1\frac{5}{8}$ yards (b) 39¢

8. $1\frac{1}{8}$ pounds

## Answers to Exercises 6-8b

1. (a) 4 (b) 100 (c) $53\frac{5}{6}$ (d) $37\frac{4}{5}$

2. (a) $7\frac{5}{8}$ (c) $17\frac{3}{32}$ (e) 75
(b) $10\frac{2}{16}$ or $10\frac{1}{8}$ (d) 18 (f) $\frac{91}{8}$ or $11\frac{3}{8}$

3. (a) $\frac{2}{32}$ or $\frac{1}{16}$ (e) $\frac{129}{21}$ or $6\frac{1}{7}$
(b) $1\frac{11}{16}$ (f) $\frac{231}{1000}$ (i) $\frac{5}{12}$
(c) $\frac{19}{14}$ or $1\frac{5}{14}$ (g) $\frac{7}{30}$ (j) $\frac{5}{12}$
(d) $\frac{113}{64}$ or $1\frac{49}{64}$ (h) $\frac{7}{30}$ (k) 16 (l) $7\frac{1}{20}$

4. (a) They are equal. (b) They are equal.

5. (a) $\frac{7}{8}$ (b) 15 (c) $\frac{ad + bc}{bd}$

6. (a) $\frac{3}{5}$ (b) 15

7. No.

8. Sums of columns, rows, and diagonals are $22\frac{1}{2}$.

9. 5¢

## Answers to Exercises 6-9

1. Mental Arithmetic
(a) $\frac{2}{4}$ or $\frac{1}{2}$ (f) $\frac{5}{16}$ (k) $\frac{4}{32}$ or $\frac{1}{8}$
(b) $\frac{1}{6}$ (g) $\frac{3}{12}$ or $\frac{1}{4}$ (l) $\frac{15}{18}$ or $\frac{5}{6}$
(c) $\frac{7}{8}$ (h) $\frac{a - c}{b}$ (m) $\frac{1}{1000}$
(d) 3 (i) $\frac{11}{35}$ (n) $\frac{1}{12}$
(e) $\frac{9}{16}$ (j) $\frac{13}{100}$ (o) $\frac{4}{9}$

2. (a) $\frac{8}{64}$ or $\frac{1}{8}$

(b) $\frac{15}{16}$

(c) $\frac{2}{9}$

(d) $3\frac{7}{15}$ or $\frac{52}{15}$

(e) $2\frac{15}{16}$ or $\frac{47}{16}$

(f) $1\frac{1}{9}$ or $\frac{10}{9}$

(g) $1\frac{1}{4}$ or $\frac{5}{4}$

(h) $14\frac{1}{3}$ or $\frac{43}{3}$

(i) $\frac{791}{1000}$ or 0.791

(j) $\frac{4}{9}$

(k) $\frac{4}{9}$

(l) They are equal.

(m) Yes.

3. Jane lives farther because $3 \cdot 16 > 5 \cdot 7$.

*4. A Mel's home

B Vic's home

C Bob's home

B is $\frac{3}{72}$ or $\frac{1}{24}$ of a mile farther than A.

C is $\frac{11}{72}$ of a mile farther than A.

5. $14\frac{3}{4}$ rods

6. $\frac{2}{5}$ second

7. $3\frac{1}{8}$ yard

8. $\frac{9}{16}$ of a bottle

9. $\frac{3}{4}$ hour or 45 minutes

10. (a) raised

(b) $\frac{1}{6}$¢ per pound

11. (a) Less

(b) If the difference were exactly 11, the larger number would be $6\frac{3}{4} + 11$ or $17\frac{3}{4}$. Since $17\frac{1}{4} < 17\frac{3}{4}$, the difference is less than 11.

12. In general, yes, unless the number subtracted is zero, or the original fraction has a value of 1.

## Sample Test Questions

### Completion

Directions: Read each statement carefully. Decide on the word, phrasé or number which best completes each statement.

| | |
|---|---|
| 15 | 1. If $\frac{1}{2} = \frac{x}{30}$, then x = _____. |
| 16/15 | 2. The reciprocal of 15/16 is _____. |
| 9/10 | 3. The rational number midway between 4/5 and 1 is _____. |
| 1/4 | 4. If $(1/2 \times 5/6) - (1/3 \div 2) = x$, then x = _____. |
| ad = bc | 5. Two rational numbers a/b and c/d are equal if and only if _____. |
| 56 | 6. The least common multiple of the denominators of 4/7 and 7/8 is _____. |
| 1 | 7. The greatest common factor of the denominators of 4/7 and 7/8 is _____. |
| 9/10 | 8. The product of 9/10 and 10/10 is _____. |
| 41/30 or $1\frac{11}{30}$ | 9. If $1\frac{1}{2} + 1\frac{1}{5} - 1\frac{1}{3} = x$, then x = _____. |
| 111/1000 | 10. If $1/10 + 1/100 + 1/1000 = x$, then x = _____. |

### Multiple Choice

Directions: Read each statement carefully. Select the word, phrase or number which makes each sentence true.

(b) 1. If $1/2 \times 1/3 = x$, then x = (a) 2/3 (b) 1/6 (c) 2/5 (d) none of these.

(a) 2. If $(1/5 \times 1/3) + 1/3 = x$, then x = (a) 2/5 (b) 2/15 (c) 16/45 (d) none of these.

(b) 3. If $\frac{4}{5} + \frac{4}{3} + 1\frac{2}{3} = x$, then x = (a) $\frac{19}{5}$ (b) $\frac{57}{15}$ (c) $\frac{13}{15}$ (d) none of these.

(b) 4. $\frac{1}{2} \div (\frac{1}{3} + 1) =$ (a) $\frac{2}{3}$ (b) $\frac{3}{8}$ (c) $\frac{1}{2} + \frac{1}{3} \div 1$

(c) 5. The identity element for the addition operation on rational numbers is (a) one (b) the reciprocal of zero (c) zero (d) none of these.

(c) 6. If $4\frac{2}{3} \div \frac{2}{3} = x$, then $x =$ (a) $\frac{28}{9}$ (b) $\frac{8}{3}$

(c) 7 (d) none of these.

(a) 7. If $\frac{2}{7} \cdot (\frac{1}{5} \div \frac{1}{4}) = x$, then $x =$ (a) $\frac{8}{35}$ (b) $\frac{1}{70}$

(c) $\frac{5}{14}$ (d) none of these.

(b) 8. The point midway between $\frac{a}{b}$ and $\frac{c}{d}$ is always (a) $\frac{ad + bc}{bd}$

(b) $(\frac{a}{b} + \frac{c}{d}) \times \frac{1}{2}$ (c) $2(\frac{ad + bc}{bd})$ (d) none of these.

(c) 9. If $\frac{a}{b} > \frac{c}{d}$, then (a) $bc > ad$ (b) $ac > bd$

(c) $ad > bc$ (d) none of these.

(c) 10. $(\frac{a}{b} + \frac{c}{d}) + \frac{e}{f} = \frac{a}{b} + (\frac{c}{d} + \frac{e}{f})$ is an illustration of the

(a) distributive property of addition (b) commutative property of addition (c) associative property of addition (d) none of these.

## True - False Questions

Directions: Indicate whether the following statements are "True" or "False".

(F) 1. A number is a rational number if it can be expressed as the quotient of two whole numbers.

(T) 2. The sum of two rational numbers is always a rational number.

(T) 3. $\frac{9}{16} \div \frac{72}{64} = \frac{1}{2}$

(F) 4. The reciprocal of a rational number is always a rational number.

(T) 5. The rational number $\frac{7}{8}$ is less than the rational number $\frac{9}{10}$ because $\frac{7}{8}$ lies to the left of $\frac{9}{10}$ on the number line.

(T) 6. $\frac{19}{20} > \frac{18}{19}$

(T) 7. If the greatest common factor of the numerator and denominator of a fraction is one, the fraction is the simplest name of a rational number.

(F) 8. $(\frac{2}{5} \div \frac{1}{2}) \div \frac{1}{3} = \frac{2}{5} \div (\frac{1}{2} \div \frac{1}{3})$

(F) 9. The numerator of the complex fraction $\frac{\frac{4}{9}}{\frac{5}{7}}$ is the divisor and the denominator is the dividend.

(F) 10. The rational number $\frac{n}{n}$ where $n \neq 0$ is the identity element for the addition operation on rational numbers.

## Matching

Directions: Select from column B the property or definition for which each item in column A is a specific instance.

Sample example:

(h) 11. $(\frac{1}{4} + \frac{1}{7}) \div \frac{3}{8} = (\frac{1}{4} \div \frac{3}{8}) + (\frac{1}{7} \div \frac{3}{8})$

Column A

(e) 1. $\frac{2}{3}(\frac{1}{2} + \frac{1}{4}) = (\frac{2}{3} \cdot \frac{1}{2}) + (\frac{2}{3} \cdot \frac{1}{4})$

(a) 2. $\frac{3}{32} + \frac{4}{16} = \frac{4}{16} + \frac{3}{32}$

(f) 3. If $\frac{2}{3} < \frac{3}{4}$ and $\frac{3}{4} < \frac{8}{9}$, then $\frac{2}{3} < \frac{8}{9}$.

Column B

(a) The commutative property of addition.

(b) The associative property of addition.

(c) The commutative property of multiplication.

| Column A (cont'd) | Column B (cont'd) |
|---|---|
| (d) 4. $(\frac{1}{2} \cdot \frac{2}{3}) \cdot \frac{3}{4} = \frac{1}{2} \cdot (\frac{2}{3} \cdot \frac{3}{4})$ | (d) The associative property of multiplication. |
| (b) 5. $(\frac{a}{b} + \frac{c}{d}) + \frac{e}{f} = \frac{a}{b} + (\frac{c}{d} + \frac{e}{f})$ | (e) The distributive property of multiplication over addition. |
| | (f) The transitive property for inequality. |

## Applied Problems

$3\frac{3}{4}$ 1. If the scale on a road map indicates 1" = 20 miles, then a distance of 75 miles would be indicated by a line segment of how many inches?

$2\frac{1}{3}$ lbs. peanuts 2. If a certain mixture formula states that there should be $\frac{1}{2}$ lb. of peanuts to each pound of walnuts, what would the number of pounds of peanuts be if you bought 7 pounds of this mixture?

$\frac{1}{3}$ 3. A family was traveling on the highway from Ann Arbor, Michigan to Detroit. They were traveling at 60 m.p.h. As they approached the exit a sign stated "reduce to $\frac{1}{2}$ speed." The sign at the exit to Detroit stated "Exit speed 10 m.p.h." What fraction of the half-speed was this?

5 4. How many packages can you wrap with $7\frac{1}{2}$ yards of wrapping paper if each package requires $1\frac{1}{2}$ yards?

$38\frac{2}{3}$ 5. The following amounts of newspapers were collected for a school paper drive. $21\frac{1}{3}$ lbs., 13 lbs., $22\frac{1}{4}$ lbs., $23\frac{2}{3}$ lbs., 14 lbs., $21\frac{3}{4}$ lbs. If you divide this amount into three equal bundles how many pounds of paper will there be in each bundle?

$1.00 6. If three yds. of material cost $2.00, what will be the price of $1\frac{1}{2}$ yds. of the same material?

7. Change the following recipe for making 8 almond cupcakes so that it can be used to make 4 cupcakes.

Recipe for 4 Cupcakes

| | | | |
|---|---|---|---|
| $\frac{1}{8}$ | cup shortening | $\frac{1}{8}$ | tsp salt |
| $\frac{1}{4}$ | cup sugar | $\frac{7}{16}$ | cup flour |

$2$ egg yolks

$\frac{1}{8}$ cup milk

$\frac{2}{3}$ tsp baking powder

$\frac{1}{4}$ tsp almond extract

Recipe for 8 Cupcakes

$\frac{1}{4}$ cup shortening

$\frac{1}{2}$ cup sugar

$4$ egg yolks

$\frac{1}{4}$ cup milk

$\frac{1}{4}$ tsp salt

$\frac{7}{8}$ cup flour

$1\frac{1}{3}$ tsp baking powder

$\frac{1}{2}$ tsp almond extract

8. The following table shows the relation of distance traveled to speed and time. Find the number that belongs in each blank space.

| | Speed mi. per hr. | Time hrs. | distance traveled miles | | Answers |
|---|---|---|---|---|---|
| (a) | 20 | $3\frac{1}{2}$ | ____ | (a) | 70 mi. |
| (b) | ____ | $\frac{1}{3}$ | 20 | (b) | 60 m.p.h. |
| (c) | 60 | $\frac{3}{5}$ | ____ | (c) | 36 mi. |
| (d) | 60 | ____ | 100 | (d) | $1\frac{2}{3}$ hr. |

(736 m.p.h.) 9. If a jet plane is traveling at the speed of sound (1080 ft/sec), how many miles per hr. is it traveling? Express your answer to the nearest mile per hr. Note: a second is $\frac{1}{3600}$ of an hour and a mile = 5280 ft.

(7) 10. Solve for x in the number sentence

$$\frac{55}{77} = \frac{5}{x}$$

# CHAPTER 7

# MEASUREMENT

## Introduction

The idea of measurement is of fundamental importance in modern life. This is true in the day by day activities of ordinary citizens and equally true in the activities associated with most vocations. Newspapers and other kinds of reading matter are full of references to measurements of one kind or another. Consequently, there is a great deal of information about the topic which could conceivably be taught in the seventh grade, and which some, if not all, seventh grade pupils would find both useful and interesting. The ability to understand what other people are doing depends to a considerable extent on understanding the kinds of things they measure and the techniques of measurement which are used. This in turn depends upon an understanding of the nature of the thing measured. Adequate treatment of many techniques of measurement is therefore more suitable content for courses in other areas, such as science, than in mathematics.

The fundamental notion of measurement, however, is common to measurement in all fields, and the development of basic concepts is therefore an important topic in the mathematics curriculum. The measurement of one, two, and three dimensional figures is also essential both for development of certain geometric concepts and for applying these concepts. Furthermore, many things which are not geometric in character are measured by relating their properties to linear or circular scales. In this chapter, therefore, attention is focused chiefly upon development of basic concepts which underlie measurements of all kinds, and upon measurements of line segments, surfaces, solids, and angles.

While pupils entering the seventh grade have had a substantial amount of experience with measurement, some of them may have no clear understanding of the basic concepts and assumptions which underlie the process. Among the concepts developed in this chapter are the following:

1. The size of a collection of discrete objects is found by counting and described by numbers; the size of a continuous quantity is found by measuring and described by a measurement which contains both a number and a unit of measure.

2. The process of counting separate objects yields a whole number which represents exactly the size of the collection; the process of measuring yields a number of units which is never exact, but is an approximate representation of the size of the quantity measured.

3. Since all measurements are approximate, a measurement should be reported so as to indicate its precision, or its greatest possible error. The precision is shown by naming the smallest unit or subdivision used. This implies the size of the greatest pos-

sible error, or the interval within which the true measurement lies. Pupils should be encouraged to report measurements which they have made in a way consistent with this idea, but cannot be expected at this stage to understand fully its implications for computed measurements.

4. As suggested above, development of a method for measuring anything rests upon understanding the aspect of it which is to be measured. This is true of the measurement of geometric continuous quantities to which a large part of this chapter refers. Therefore, four assumptions regarding the nature of geometric continuous quantities are stated.

5. A unit of measure for geometric quantities must be of the same kind as the thing to be measured--a unit line segment to measure a line segment, a unit surface to measure a surface.

6. The size of a unit of measure is arbitrary.

7. The measure of a geometric quantity is obtained by subdividing it into parts the size of the unit, and counting the number of unit parts.

8. Standard units of measure are necessary in highly organized societies, and are established by social agreement.

9. It is convenient to have related units for length, surface, and volume, such as the inch, square inch, and cubic inch.

10. The area of the interior of a simple closed curve, such as the rectangle, may be found by computation from lengths rather

than direct measurement of surface; the same is true for the volume of the interior of a rectangular solid.

11. Just as there are many names for the same number, there are many names for the same measurement.

### 7-1. Counting and Measuring

The purpose of this section is to call attention to the difference between the process of counting and the process of measuring, and to the kind of quantity to which each is applicable.

### Answers to Exercises 7-1

1. Counting
2. Measuring
3. Counting
4. Measuring
5. Counting
6. Measuring
7. Measuring
8. Measuring
9. Yes
10. Yes
11. No
12. No
13. Yes
14. No

15. Yes

16. Yes

## 7-2. Properties of Continuous Quantities

Measuring geometric continuous quantities is accomplished by the process of subdividing the quantities into unit parts. The purpose of Section 7-2 is to direct attention to four assumptions about the nature of geometric quantities on which the process of measuring depends. Property 1 states, "A geometric figure may be moved without changing its size or shape." To help pupils understand this property it would be well to suggest some illustrations which do not exhibit this property. For example, if a rubber band is placed around a box, both size and shape of the band will be altered. When you mold clay, the shape of the clay changes, but not the amount of clay. When a picture slide is projected on a screen, the sizes of the objects in the picture change, but not the shapes. We think of geometric figures as keeping both size and shape when they are moved. This is important for comparison of two geometric quantities, and also for subdividing a quantity into parts.

In Property 2 there is the hidden assumption that, for two geometric quantities $a$ and $b$, $a = b$, or $a$ is greater than $b$, or $a$ is less than $b$, and that exactly one of these relations is true. Property 2 then asserts that by using Property 1 it is possible to determine which of these relations is true for any two quantities of the same kind.

Property 3 is basic to the notion that differently shaped polygons, or other simple closed curves, may have perimeters of the same length or interiors of the same area. It leads to Property 4, which states the assumption that a quantity may be subdivided without producing a change in the size of the whole. Both of these properties are used in the process of measuring.

Before beginning Section 2 it would be well to review the concepts of point, line, ray, line segment, and simple closed curve, which were introduced in Chapter 4.

The ideas of Section 2 are developed mainly through exercises. In Set 7-2a the questions should probably be used for class discussion. In Set 7-2b, Problems 1-3 relate to comparison of line segments and comparison of interiors of simple closed curves. In Problems 4-8 it is necessary to cut up one of the figures and rearrange the parts in order to make a comparison. Problems 6-8 direct attention to the possibility that closed curves of the same length may enclose different areas, and that the same area may be enclosed by curves of differing length.

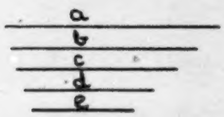
The paragraphs on Subdividing Continuous Quantities illustrate with line segments the essential notion that a unit segment "n" may be laid off repeatedly on a given segment to locate a point such that the segment remaining is less than one unit. Exercises 7-2c develop this notion further. These problems provide the basis for the definition of the terms "measure" and "unit of measure." Note that, according to the definition, the measure is a number, while the length is defined as a phrase which includes both the measure and the unit.

## Answers to Exercises 7-2a

1. Crack between blackboard sections, edge of window pane, etc. Have pupils indicate endpoints of each segment. (Note that "continuous quantity" does not mean one thought of as continuing indefinitely like the geometric concept of line, but rather that within its boundary it is unbroken.)
2. Surface of blackboard section, surface of window pane, etc. Have pupils indicate the simple closed curves as well as their interiors.

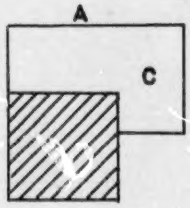
## Answers to Exercises 7-2b

1. (a) Same
   (b) $\overline{RT}$
   (c) Same
   (d) $\overline{RT}$
   (e) $\overline{RT}$
2. (a) a is longer than d.
   c is longer than e.



3. (a) Figures may be placed in a variety of ways, e.g.

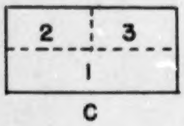
C is the smallest.

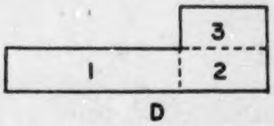


(b) A > C; B > C; D > C

4. (a) No, their shapes are different.

(b) One way:





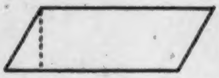
(c) The areas of the interiors are the same size.

5. (a) All are approximately equal.

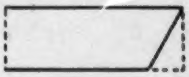
(b) To compare A with B, cut along shorter diagonal of A.

" " A with C, turn A over.

" " A with D, cut as follows:



and reassemble as follows:

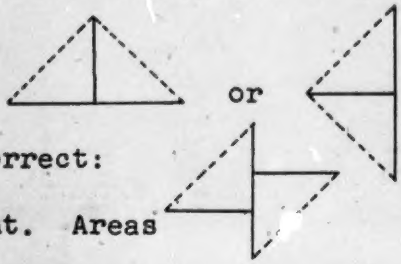


(c) All should have the same area.

6. (a) Curves have the same length.

(b) Interior of E is larger than interior of F.

7. Pupils will probably assemble the parts like this:



Equally correct: or any other rearrangement. Areas of new figure and area of the square are the same.

8. (a) Note that two sides of B have the same lengths as two sides of A.

   (b) Various rearrangements are possible

   (c) Interiors have the same size.

   (d) Pupils will probably draw a variety of curves, some having the same length, some having different lengths. All interiors are the same size, however, if A and B are placed together so they do not overlap.

## Answers to Exercises 7-2c

1. Length of $\overline{AB} \approx 4c$

2. (a) Size of interior of A $\approx$ 3 times size of interior of B.

   (b) 3

3. (a) Length of curve D $\approx$ 2 times the length of curve C.

   (b) Interior of curve D $\approx$ 3 times interior of curve C.

4. 3

5. (a) 4

   (b) 3

6. (a) DE, AF, FC are approximately the same length.
   AD, DB, EF are " " " "
   BE, EC, DF are " " " "

   (b) ADF, FEC, DBE, DEF, ADEF, FDEC, DBEF, ADEC, DFCB, AFEB, ABC.

6. (c) $\triangle ADF$, $\triangle DEF$, $\triangle DBE$ all have the same area as $\triangle EFC$. The interior of $\triangle ABC$ is 4 times the interior of $\triangle EFC$.

   (d) Interiors of ADF, EFC, DEF, DBE have the same area.
   Interiors of ADEF, FDEC, DFEB have the same area.
   Interiors of ADEC, FDBC, and EFAB have the same area.

## 7-3. Measuring

Note that in this paragraph distinction is made between the meanings of the terms "measure", "unit of measure", and "measurement" The "measure" is a number, the "unit of measure" is the arbitrary segment (or surface or solid) with which the quantity is compared, and the "measurement" includes both the measure and unit to give an adequate description of size. The set of Exercises 7-3a serves to familiarize the pupils with these meanings. The paragraphs which follow and Exercises 7-3b serve to emphasize the important characteristics of a unit of measure.

### Answers to Exercises 7-3a

1. (a) Segment of length c.

   (b) 4

2. (a) Interior of curve B.

   (b) 3

3. (a) 6t

   (b) 12t

(c) Length of curve D is 2 times the length of curve C.

(d) Interior of curve C is unit; interior of curve D is 3 times interior of curve C.

4. Unit of measure was interior of box B; interior of box A is 3 times interior of box B.

## Answers to Exercises 7-3b

1. Answers will vary but should be reasonable.
2. " " " " " " "
3. Differences arise from differences in unit of measure, i.e., length of foot.
4. Width, thickness; width, thickness.
5. Book, chalk box, ream of paper, etc.
6. The measure is the number of sheets used but the size is the number and the name, "note-book sheets."
7. Measures varied.

The paragraphs relating to cutting up units of measure continue to emphasize that the unit used to measure a surface must be a surface. The term "area" is used for the first time, in a way which is analogous to the way in which the term "length" is used with reference to segments.

Assumption 4 is used again in two ways: (1) to subdivide the rectangle into parts, each part having the same size as the unit of measure; and (2) to subdivide the unit of measure in ways which make it possible to cover the entire interior of the rectangle. Assumption 3 is used in assigning as the measure of the rectangle the combined measure of all the units and subdivisions employed in the covering process, and also in verifying that the units may be laid on in any position chosen without changing the measure of the rectangle.

The purpose of Exercise 7-3c is to develop further the concept of area, to illustrate the fact that any unit surface may be used to measure area, and to show also that the square unit usually used (square inch, square foot, etc.) is probably as convenient a choice as could be made.

### Answers to Exercises 7-3c

1.

2. The answer depends upon the size of sheet used. For an $8\frac{1}{2}$ × 11 sheet of paper, the computed numbers of units required are as follows:

   Rectangle -- about 47

   Isosceles right triangle -- about 47

   Equilateral triangle -- about 54

   Circle -- about 30

3. The circle is probably hardest to use because of the space between the circles.

4. Same for rectangles (2" × 1") and isosceles right triangle

with 2" legs.

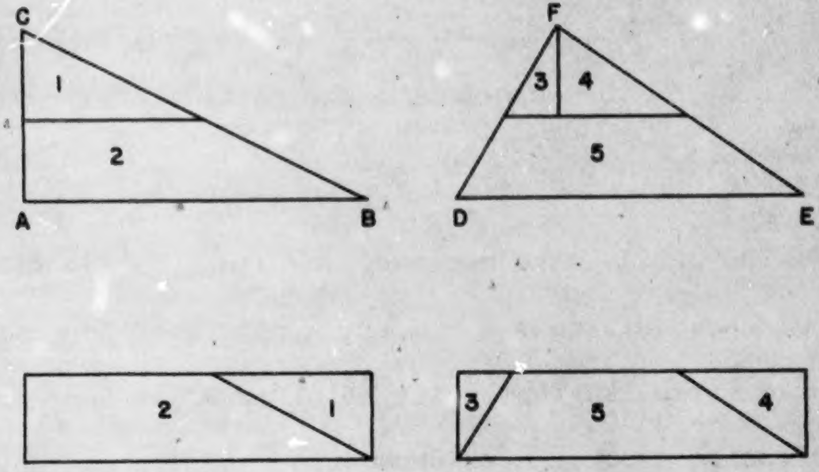
5.

6.

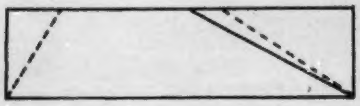
7. Balls are poor unit because of space between the balls. Rectangular blocks would be much better.

8. BRAINBUSTER

(a)

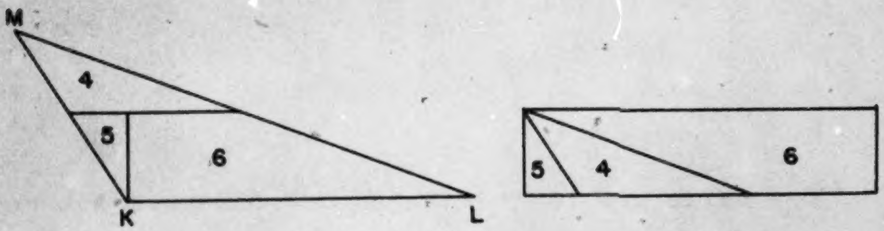


(b)

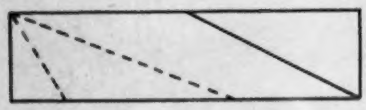


The solid line shows the arrangement of the parts of triangle ABC, and the dashed lines the arrangement of the parts of triangle DEF.

(c)



(d)



The solid line shows the parts of triangle ABC, and the dashed line the parts of triangle MKL.

## 7-4. Standard Units

The pupils have had many exercises to demonstrate the meaning of measurement and the arbitrary nature of the unit. This section should establish the social need for units that are the same for the entire group. Mass production and the convenience of interchangeable parts provide a wealth of material for pointing out this need. The historical material should be of interest and show society's increasing need for standard units.

Archimedes used density in determining the purity of gold. By experiment, he found that the same weights of silver and of gold displaced different volumes of water. By immersing gold that had the same weight as the crown and comparing the water displacement with that displaced by the crown, Archimedes proved that other metals were used to replace some of the gold.

The metric system was omitted from this chapter only because of the chapter's length; it will be discussed in Chapter 8. It should be stressed that our linear units are defined, by law, in terms of the metric units. The metric system is legal in the U.S. If class time permits, a discussion of the new definition of a meter as 1,650,763.73 times the wave length of orange light from krypton 86 might be interesting. This standard for the meter is difficult to visualize, but has the advantage that it can be reproduced in any good scientific laboratory and is more precise than the platinum and irridium bar in France that is the present standard for the meter. This definition has been adopted by the Advisory Committee on the Definition of the Meter and, in all probability, will be adopted by the International Conference on Weights and Measures in 1960. Current Science for the week of September 29-October 3, 1958 has an article on this at the student's level.

The approximate nature of measurement has been pointed out throughout the chapter. Continuing emphasis on the fact that measurements are not exact should be made, although this topic will be treated more formally in Section 5. Comparing measurements of the same object made by different students and also measures determined by instruments marked with varying degrees of precision help develop the concept of the approximate nature of measurement.

Some 7th grade students have difficulty reading a ruler. Through making cardboard rulers and studying the increasingly fine divisions in Figure 7-4, it is hoped that the poor students will overcome their difficulties while the better student is seeing the relationship with base 2. The "Class Problems, 7-4" should be done orally in class. Some of the actual measuring should also be done in class so that the teacher can identify and work with students who have not mastered the use of the ruler.

In the hodge-podge that is the English system of measures, there is a variety of standard units. Conversions from one unit to another cause a great deal of trouble both in mathematics and in science. Conversions are treated as number sentences with a stress on the relation between the units involved in the conversion. This should be developed in class and class practice should be provided to make sure the students understand what they are doing. The use of estimates and common sense are important aids in converting units. Measurements recorded in different units but treated as the same unit are the basis for many student errors. Pupils should be taught to read the names of units as well as the number of these units. Common sense is called on again to decide which would be the better of these units to use for a particular problem.

## Exercises 7-4a

These exercises are designed to show the value of standard units to social living. Some of the questions are answered best by personal interview with local merchants. All values to the students could be realized by having each student do one or two of the exercises.

1. Most bread is sold by weight. Loaves are baked in various sizes.
2. The size of food cans is related to volume. Since a packer could leave space at the top of a can, the law requires that each can have the weight of the contents printed on it. Students might check different brands of the same item to see if the contents of the same size can always has the same weight.
3. A few items to suggest if the students fail to find others are: bicycles - diameter of the wheel; hats - diameter of head for men and perimeter for women; pans - capacity; screws - length and threads per inch; fishing rods - length; drill bits - diameter.

### Exercises 7-4-b

These exercises give the student practice in measuring and they develop the need for fine subdivision. It is a good preparatory exercise for those who have difficulty with reading a ruler.

1. and 2. Cardboard rulers can be checked with commercial rulers.

3. (a) $2\frac{1}{2}$" (b) $3\frac{1}{2}$" (c) 2" (d) 4" (e) 3"

4. (a) $2\frac{1}{2}$ (b) $3\frac{1}{4}$ or $3\frac{1}{2}$ (c) $2\frac{1}{4}$ (d) $3\frac{3}{4}$ (e) 3 or $3\frac{1}{4}$

5. (a) $2\frac{1}{2}$ (b) $3\frac{3}{8}$ (c) $2\frac{1}{8}$ or $2\frac{1}{4}$ (d) $3\frac{3}{4}$ or $3\frac{7}{8}$ (e) $3\frac{1}{8}$

6. Answers will vary but should approximate those above.

7. 8ths of an inch since the divisions come closer to matching the line segments.

## Class Problems 7-4

1. The need to measure segments smaller than an inch.
2. Into 2 parts
3. Each section of the inch on the left of any given inch (except the 6th inch) is divided into two parts.
4. Each section is divided into two parts.
5. 4; 8; 16. (The third inch lies between 2 and 3).
6. The line segments are longer.
7. Longer lines help group the shorter lines so that the number and size of the spaces are easier to see.
8. Theoretically, there is no limit to the number of divisions.

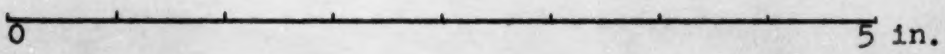
## Exercises 7-4-c

1. 32

2. (A) $\frac{3}{4}$" (B) $1\frac{5}{8}$" (C) $2\frac{7}{8}$" (D) $3\frac{3}{8}$" or $3\frac{1}{2}$" (E) $4\frac{3}{8}$"
(F) 5" (G) $5\frac{5}{8}$"

3. (a) $\frac{7}{8}$" (b) $2\frac{1}{8}$ (c) $\frac{1}{2}$" or $\frac{5}{8}$" (d) $1\frac{1}{2}$ or $1\frac{5}{8}$ (e) $\frac{5}{8}$"

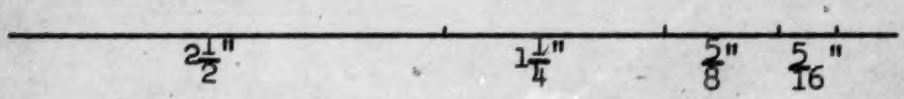
4. (a)



(b) 8

(c) The divisions on the line picture the meaning of $5 + \frac{5}{8}$; there are 8 pieces.

5. (a)



(b) $4\frac{11}{16}$ (c) $4\frac{11}{16}$

6. (a) $\frac{7}{16}$" (b) $1\frac{9}{16}$ (c) $5\frac{1}{2}$ (d) $2\frac{1}{2}$"

(e) $\frac{3}{4}$ " (f) $3\frac{1}{2}$" (g) $\frac{1}{2}$" (h) $\frac{5}{8}$"

7. (a) AB $\approx \frac{7}{16}$" FG $\approx$ 1"

BC $\approx \frac{13}{16}$" GH $\approx \frac{1}{2}$"

CD $\approx \frac{3}{16}$ HI $\approx \frac{5}{8}$"

DE $\approx \frac{9}{16}$" IJ $\approx \frac{5}{8}$"

EF $\approx \frac{3}{4}$"

(b) There is a possibility of a slight ($\frac{1}{16}$") discrepancy due to the approximate nature of measures.

8.

| Base 10 | | Base 2 |
|---|---|---|
| $\frac{1}{2}$ | = | $\frac{1}{10}$ |
| $\frac{1}{4}$ | = | $\frac{1}{100}$ |
| $\frac{1}{8}$ | = | $\frac{1}{1000}$ |
| $\frac{1}{16}$ | = | $\frac{1}{10,000}$ |

9. Since all divisions are obtained by dividing existing divisions by 2, the number of sections increases by powers of

2, or, in base 2 notation, by adding a zero to show the number of divisions.

## Answers to Exercises 7-4-d

1. Answers will vary.

2. Classrooms, halls and other lengths can also be measured. It would be wise for the teacher to have a good idea of the measures assigned.

3. Answers will vary for each school.

4. (a) $2\frac{1}{2}$" (b) $3\frac{3}{8}$" (c) $2\frac{3}{16}$" (d) $3\frac{13}{16}$" (e) $3\frac{1}{8}$

5. (a) $7\frac{1}{2}$" (b) 7" (c) 9" (d) $7\frac{3}{4}$" (e) $6\frac{13}{16}$ (f) $7\frac{3}{4}$

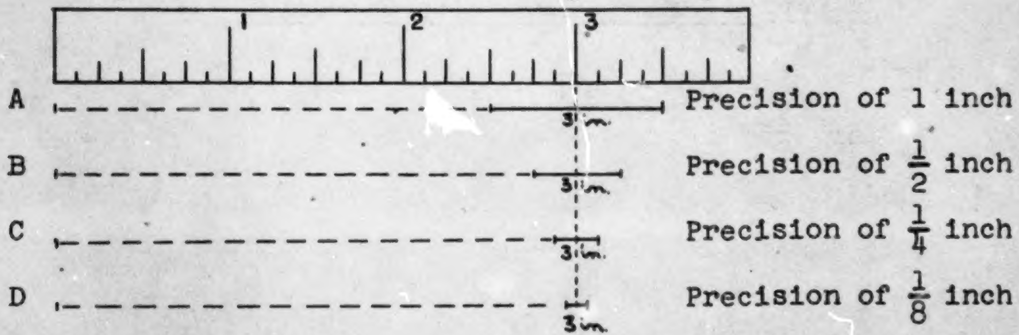
6. (a) $9\frac{7}{16}$" (computed) (b) $\approx 9$" (c) $\approx 9\frac{1}{2}$" (d) $\approx 10\frac{1}{2}$"
(e) $\approx 8$" (f) $\approx 6$"

7. (a) Length $\approx$ 5400 in. (Answers in feet are acceptable.
Width $\approx$ 900 in. See part b).
Height $\approx$ 540 in.
(b) Length $\approx$ 450 ft. $\approx$ 150 yd.
Width $\approx$ 75 ft. $\approx$ 25 yd.
Height $\approx$ 45 ft. $\approx$ 15 yd.

*8. The S. S. United States is 990 ft. long. The dimensions given for the Ark make it considerably larger than it would seem that the engineering of the day permitted. The length is greater than the length of a football field.

## 7-5. Precision and Greatest Possible Error

This section is the first attempt in this book to formalize the approximate nature of measures. Precision is introduced through work with the ruler; the student is asked to develop a definition for precision. It is important that the final definition decided upon in class convey an accurate concept. The concept can be developed rather easily through exercises. It is much more difficult to verbalize the concept of precision. The question in the student's text is designed to stimulate thinking rather than being an example that has a right or wrong answer. Precision, in linear measure, is the total length of the segment on which the endpoint of the measured segment must fall in order that the length of this segment will have a given name. To help make this clear, the drawing below shows a series of 3-inch segments. Segment A shows the range when the precision is 1 inch; B, when the precision is $\frac{1}{2}$ inch; C, when the precision is $\frac{1}{4}$ inch; and D, when the precision is $\frac{1}{8}$ inch. The broken section shows the entire segment while the solid segment at the right shows the range of the points where the endpoint of the measured segment may fall for each degree of precision.


1
2
3
A
3 in.
Precision of 1 inch
B
3 in.
Precision of $\frac{1}{2}$ inch
C
3 in.
Precision of $\frac{1}{4}$ inch
D
3 in.
Precision of $\frac{1}{8}$ inch

As the material has been developed, this has been related directly to the size of the divisions on the measuring instrument, but the skill of the person using the instrument is also a factor in keeping the range of the measurement within the limits of the indicated precision.

The concept of the greatest possible error is directly related to precision. It is the error permitted in one direction from the point that names the measure and is, therefore, one-half of the indicated precision.

Two notations for indicating the degree of precision are presented. The first reports a measure in fractions which indicate the smallest subdivision used - $\frac{1}{4}$, $\frac{1}{2}$, etc. The second is shown by the permissible deviation from the name of the measure stated.

Approximation in measures is a new and difficult concept for 7th graders. The treatment is intended to be groundwork for future understanding. Enlarged drawings of inches and their subdivisions on the chalkboard as well as practice with the material is needed.

## Answers to Class Discussion Exercises 7-5-a

1. $3\frac{1}{2}$"
2. $3\frac{1}{4}$" or $3\frac{1}{2}$"
3. $3\frac{3}{8}$"
4. $3\frac{3}{8}$"

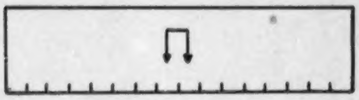
5. Line (c) 2"; $2\frac{1}{4}$"; $2\frac{1}{8}$" or $2\frac{1}{4}$"; $2\frac{3}{16}$"

6. In each case there is at least one point at which the measure may stay the same. A higher degree of precision may produce further change, if such rulers are available.

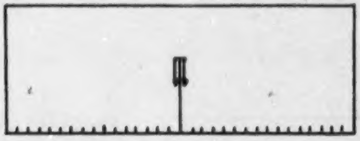
### Answers to Class Discussion Exercises 7-5-b

1. Precision, in linear measure, is the total length of the segment on which the endpoint of the measured segment must fall in order that the length of this segment will have a given name.

2. $\frac{1}{16}$"

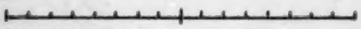


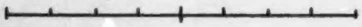
3. $\frac{1}{32}$"



### Answers to Exercises 7-5-a

1. $\frac{1}{2}$

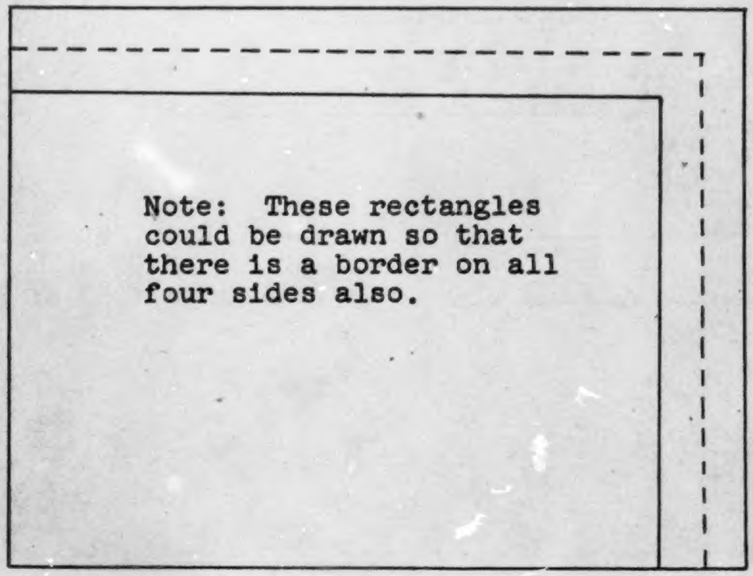
2.  (Divided into 8ths of an inch)

3.  (Divided into 4ths of an inch)

4.

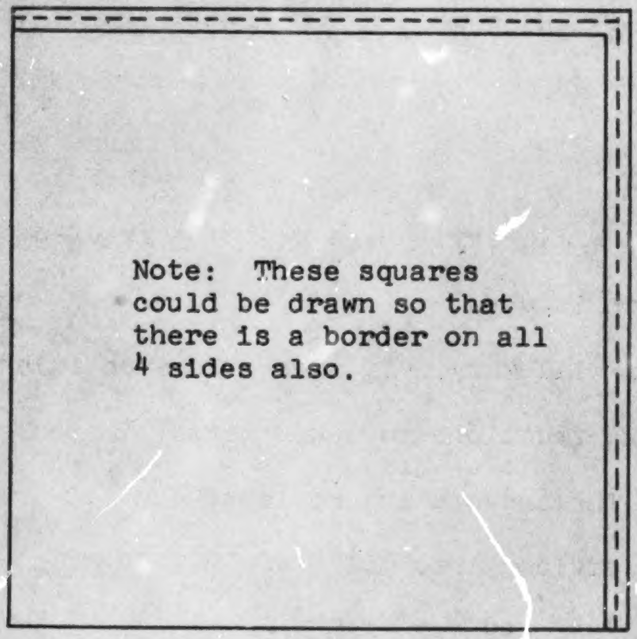
| Figure | | (1) | | (2) | | (3) | |
|---|---|---|---|---|---|---|---|
| | | 1/2" marks | Error | 1/8" marks | Error | 1/16" marks | Error |
| (a) | l<br>w | 3"<br>1 1/2" | 1/4" | 2 3/4" or 2 7/8"<br>1 3/8" | 1/16" | 2 13/16"<br>1 3/8" or 1 6/16" | 1/32" |
| (b) | l<br>w | 2"<br>2" | 1/4" | 2" or 2 1/8"<br>2" or 2 1/8" | 1/16" | 2 1/16"<br>2 1/16" | 1/32" |
| (c) | l<br>w | 1 1/2" or 2"<br>1" | 1/4" | 1 3/4" or 1 6/8"<br>7/8" or 1" | 1/16" | 1 3/4" or 1 12/16"<br>15/16" | 1/32" |
| (d) | l<br>w | 2 1/2"<br>1/2" | 1/4" | 2 5/8"<br>1/2" or 5/8" | 1/16" | 2 5/8" or 2 10/16"<br>9/16" | 1/32" |

5. (a), (b) and (d)



Note: These rectangles could be drawn so that there is a border on all four sides also.

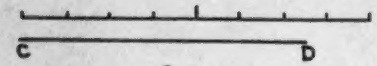
6. (a) (b)



Note: These squares could be drawn so that there is a border on all 4 sides also.

## Answers to Exercises 7-5-b

1.



$CD = 1\frac{3}{4}''$

2. $CD = 1\frac{3}{4} \pm \frac{1}{8}''$

3. $CD = 1\frac{3}{4}''$

4. Between $1\frac{5}{8}''$ and $1\frac{7}{8}''$ ; $\frac{1}{8}''$

5. (a) $8^{th}$

   (b) $1\frac{3}{16}''$ and $1\frac{5}{16}''$

   (c) $\frac{1}{16}''$

   (d) $1\frac{1}{4} \pm \frac{1}{16}$ in.

6. (a) 8th

   (b) $2\frac{3}{16}$" and $2\frac{5}{16}$"

   (c) $\frac{1}{16}$"

   (d) $2\frac{2}{8}$"

7. Answers will vary. Be sure the ± greatest possible error notation is used.

8. Answers will vary. Be sure precision is shown by not changing fractions to lower terms.

9. Note: The answers in Problems 9 and 10 are given for rulers divided into 16ths of an inch.

   AB = $4\frac{3}{16}$" and $4\frac{3}{16}$" $\pm \frac{1}{32}$"

   BC = $2\frac{12}{16}$" and $2\frac{3}{4} \pm \frac{1}{32}$" or $2\frac{11}{16}$" and $2\frac{11}{16} \pm \frac{1}{32}$"

   AC = $6\frac{0}{16}$" and $6 \pm \frac{1}{32}$"

10. AB = $3\frac{0}{16}$" and $3 \pm \frac{1}{32}$"

    BC = $1\frac{13}{16}$" and $1\frac{13}{16} \pm \frac{1}{32}$"

    CD = $5\frac{4}{16}$" and $5\frac{1}{4} \pm \frac{1}{32}$"

    AD = $2\frac{10}{16}$" and $2\frac{5}{8} \pm \frac{1}{32}$"

## 7-6. Measurement of Angles

In Chapter 4 the pupils have been introduced to the concept of angle as the set of points on two rays with a common endpoint, and have learned to describe the position of a point as being on the angle, in the interior of the angle, or in the exterior of the angle.

The measurement of angle follows essentially the same ideas as measurement of a line segment; that is, (1) the unit for measuring an angle must be itself an angle; (2) the interior of the angle is subdivided by drawing rays which form angles like the unit angle; (3) the measure of the angle is the number of unit angles into which it is subdivided. After these ideas are developed in Exercises 7-6a, the standard unit, the degree, and the scale for measuring angles in degrees, are introduced. The unit angle is determined by a set of 360 rays with a common endpoint, which divide the plane into 360 angles of the same size, each of which is an angle of one degree. The scale is established by numbering the rays in order from zero to 360. Figure 7-6a shows such a scale.

The most common type of protractor shows the scale only from 0 to 180, and usually contains two such scales, one with 0 at the right end and the other with 0 at the left end of the semicircular scale. Each pupil should have a protractor and should become proficient in using it, both to measure a given angle and to draw an angle of a specified number of degrees.

The angle is defined as having a measure greater than 0 degrees and less than 180 degrees. The exterior measure of an angle is found by subdividing the exterior into unit angles. Since the complete plane is thought of as divided into 360 angles, each of one degree, the exterior measure can also be found by subtracting the measure of the angle from 360.

Angles are classified as acute, right, or obtuse, according to their measures. The pupils should practice estimating the number of degrees in an angle by comparing it with a right angle.

Perpendicular rays are defined as rays which form a right angle and the term "perpendicular" is also applied to lines, segments, and to combinations of these.

## Answers to Exercises 7-6a

1. The size of $\angle RST \approx 3$ u
2. " " " $\angle ABC \approx 5$ u
3. " " " $\angle EDF \approx 4$ u
4. " " " $\angle KML \approx 7$ u

5.(a) Exterior measurement of $\angle RST \approx 15$ u 6. (b) 18 u

(b) Exterior measurement of $\angle KML \approx 11$ u 7. (b) 18 u

## Answers to Exercises 7-6b

1. $48^\circ$
2. $97^\circ$
3. $90^\circ$
4. $80^\circ$

## Answers to Exercises 7-6c

1. (a) 15 (b) 35 (c) 60 (d) 90 (e) 100 (f) 25
   (g) 60 (h) 145 (i) 40 (j) 85 (k) 140 (l) 65
2. The size of $\angle A \approx 50^\circ$

The size of $\angle B \approx 90^\circ$

" " " $\angle C \approx 123^\circ$

" " " $\angle D \approx 74^\circ$

3. Exterior measure may be found by measuring to 180 on scale, then turning the protractor to measure remaining part of exterior and adding results. Exterior measures are as follows:

   $\angle A$, 310 $\qquad$ $\angle C$, 237

   $\angle B$, 270 $\qquad$ $\angle D$, 286

4. The size of $\angle BAC \approx 35$.

6. Subtract the measure of the angle from 360 to find its exterior measure.

   (a) 340 (b)315 (c) 299 (d) 270 (e) 230 (f) 181

7. 180

Answers to Exercises 7-6d

1. (a) $\angle B$, $\angle E$

   (b) $\angle A$, $\angle C$

   (c) $\angle D$, $\angle F$

2.

3. (a) 0 , 90

   (b) 90, 180

4. (a) Obtuse angles: $\angle BAF$, $\angle BAG$, $\angle BAH$

Acute angles : $\angle BAC$, BAD

Right angles : $\angle BAE$

(b) Acute angles : $\angle EAD$, $\angle EAC$, $\angle EAF$, $\angle EAG$, $\angle EAH$

Right angles : $\angle EAB$, $\angle EAK$

Obtuse angles: None

(c) Right angles : $\angle KAE$

Obtuse angles: $\angle KAD$, $\angle KAC$

Acute angles : $\angle KAH$, $\angle KAG$, $\angle KAF$

5. (a) $\angle ABC$ is acute $\qquad$ $\angle JML$ is obtuse

$\angle EDF$ is a right angle $\qquad$ $\angle PRN$ is a right angle

$\angle HKG$ is acute $\qquad$ $\angle QTS$ is obtuse

(b) See (c)

(c) The size of $\angle ABC \approx 10$

" " " $\angle EDF \approx 90$

" " " $\angle HKG \approx 65$

" " " $\angle JML \approx 120$

" " " $\angle PRN \approx 90$

" " " $\angle QTS \approx 130$

## Answers to Exercises 7-6e

1. (a) 3

(b) 5 (3 and 1 could be so interpreted also)

(c) 2

(d) 1

(e) 6

(f) 4

2. Edges of book, edges of intersecting sidewalks, intersecting edges on a rectangular prism, etc.
3. Edges of gable roof, diagonal parking line and curb, hands of clock in many positions, etc.

### 7-7. Rectangle

The two primary aims of this section are to develop the pupils' space perception, especially as it relates to perimeters and areas, and to develop the methods of finding these quantities for a rectangle by computation when the length and width are known. In development of the perception and a geometric feel for size, it is important that models be constantly in sight. Perhaps drawings of the actual sizes of the square inch, square foot, and square yard could be kept always on the blackboard as soon as area has been introduced. Or possibly models could be placed on the bulletin board. Also emphasis on estimating lengths and areas will help to develop a feeling for size. As often as possible have estimates made of lengths and areas and then have them actually measured (or computed) to check the accuracy of the estimate. This emphasis will necessarily come largely from the teacher, as such questions can only be suggested occasionally in the text. It is interesting that most people are quite surprised to see the actual size of a square yard.

Two points of terminology should be clarified.

1. By its definition a rectangle is a collection of segments. Such a set of points does not properly have an area (or possibly it should be said to have area zero). The area with which we are concerned is generally the area of the *interior* of the rectangle. In order to emphasize this, it is the plan in the text always to speak of the *area of the interior of a rectangle*, rather than the area of the rectangle. While the student may eventually lapse into the mathematical slang of speaking of the area of a rectangle, it has been felt better not to expose him to it at this point.
2. It has been emphasized in previous sections that the measure of a quantity is the number of units it contains. However in describing the result of a measurement it is meaningless to give the number without specifying the unit. Thus when we speak of a length, or an area, or a volume we shall mean the number *together with the unit*, as a length of 5 ft. or an area of 5 square inches. On the other hand we should keep clearly in mind that wherever literal symbols are introduced, they stand only for *numbers*. We add and multiply *numbers*, not units. Thus if we consider a

rectangle whose length is *5 in*. and whose width is *3 in*., and if we wish to use the notation of the number sentences $p = 2(\ell + w)$ and $A = \ell w$, we write

$$\ell = 5,\ w = 3,\ p = 2(5 + 3) = 16,\ A = 5\cdot 3 = 15$$

so we conclude that *the perimeter is 16 in*. and *the area is 15 sq. in*. Note that we do *not* write $\ell = 5$ in. or $A = 15$ sq. in. Here again, the student may in the future, without permanent harm to his character, write such mathematical slang as $A = 15$ sq. in., but at this point it has seemed better to distinguish clearly between a *length*, which requires specifying a unit, and the *number* of units $\ell$ in this length.

The question about the relation of squares to rectangles in the first paragraph of this section has two purposes. The discussion should first serve to clarify any doubt as to exactly what a square is. Then, bringing out that such a figure is a special case of a rectangle gives a good example of one set contained in another. The question in the second paragraph is intended to bring out that a segment is measured by a number line of segments and our model of this is a ruler. However, the interior of a rectangle, not being a segment cannot be measured by segments, but must be measured by the interior of a curve - something of the same kind. Some pupils may argue that you *could* measure the interior of a rectangle with a ruler by seeing how many times the ruler could be fitted in it. In a sense this is true, but point out that, if you do that, you are no longer considering the ruler as a model of a number line and using just its numbered edge. You would actually be using it as the interior of a simple closed curve.

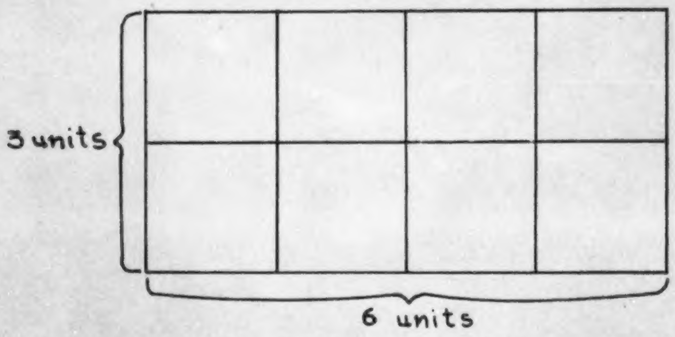
### Exercises 7-7a

Problems 1-5 are part of the developmental work and should definitely be used, although some or all of them might well be done in class. The other problems in the set are practice exercises on understanding perimeter, computation, and change of units. Problem 10 should be included in the assigned work as these ideas often seem troublesome.

The question in the first paragraph under areas about an instrument to use directly in measuring areas should provoke useful discussion. Someone may well point out that you could have a network of squares (or whatever figure you use as unit of area) marked on transparent plastic and just place this on the interior of the rectangle being measured. This is quite true, and is both an interesting and valuable idea. However, it should be pointed out that in using this you would actually have to count up the squares (and parts of squares). You cannot number the squares ahead of time so that you can just look at the number attached to a "last" square and say that is (approximately) the desired area. The difficulty is that while all segments have the same shape in the sense of being fitted by a number line (ruler), areas may occur in all sorts of shapes so that no prearranged pattern can be designed to fit all the shapes to be measured in such a way as to read off areas at sight. Your pupils may be interested to know that there are instruments which can be run around the boundaries of a simple closed curve and give at once an approximate value for the area of the interior. One such instrument is the planimeter, but its theory is far beyond the level of our work here.

In developing the method for computing areas of rectangles from the length and width it is noted that different ways of counting the unit squares of area illustrate the commutative property of multiplication.

It is hoped that class discussion will bring out the advantage of having a unit of area which is a square, one unit of length on a side. In the case of the illustration of the rectangle 6 units by 3 units, notice the effect of using, as unit of area, a square $1\frac{1}{2}$ units on a side. There is no trouble about covering the rectangle with these unit squares, as shown below.



However, the number of squares in each row is no longer the number of linear units in the length (6), and the number of rows is no longer the number of linear units in the width (3). Thus we would lose the relationship of finding the number of square units of area by multiplying the numbers of linear units in the length and width. It certainly would be possible to devise methods of computing the number of these new square units of area, but it would certainly also be more complicated.

## Exercises 7-7b.

Problems 1-5 are developmental, leading to the method for computing area of the interior of a rectangle from the numbers of linear units in the length and width. In connection with Problem 6 notice the comments in the first paragraph of the commentary in the section about emphasizing space perception. Estimating areas of doors, windows, blackboards, etc. in the various units of area and then measuring to check estimates is very valuable experience. Following Problem 6, which furnishes the information necessary, many of the problems call for conversion between various units of area. The process is the same as for linear units, but it may be well to illustrate it for the class. For example, since 1 sq. ft. = 144 sq. in., then 1 sq. in. = $\frac{1}{144}$ sq. ft. Thus to change 360 sq. in. to square feet we write 360 sq. in. = $360 \cdot 1$ sq. in. = $360 \cdot \frac{1}{144}$ sq. ft. = $\frac{360}{144}$ sq. ft. = $2\frac{1}{2}$ sq. ft.

Problems 7, 8 are designed to clarify the common confusion between the 3 inch square and the area of 3 square inches, and to emphasize the different possible shapes an area of 1 square inch may take. Problems 14-18 are a connected group and are extremely important. They emphasize the effect of doubling the dimensions of a rectangle either separately or simultaneously. Problems 14-17 approach the problem geometrically and it should be emphasized that such doubling amounts to laying identical rectangles end to end or side by side. The dotted lines in the figures in the answer key indicate the relationships we hope the pupil will see. Problem 18 shows how these geometric facts are related to properties of rational numbers that have already been studied.

## Exercises 7-7c

Problems 1-6 are drill problems in finding areas and perimeters and in converting units, with emphasis that such answers must generally be considered approximate. This section could provide further opportunity for estimating and verification. Problems 8 and 9 illustrate finding areas as a difference of known areas. Problems 9-11 are a unit which develop the idea of precision and greatest possible error in a computed measurement. They are important if the class is capable, though not easy. The length and width of a rectangle are given with a precision of $\frac{1}{4}$ inch (i.e. measurements to the nearest quarter inch). The pupil is asked to compute the area from these figures. This proves to be $8\frac{15}{16}$ sq. in. He then draws the largest and smallest rectangles which are represented by the given measurements with this precision and computes their areas. The largest possible difference between the true area and the computed area above proves to be $\frac{49}{64}$ sq. in. so the result is written in the form

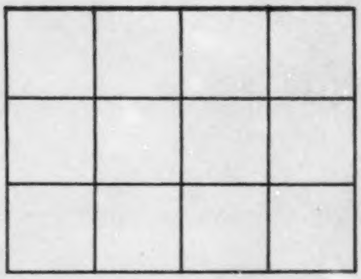
$$\text{Area is } \left(8\frac{15}{16} \pm \frac{49}{64}\right)\text{sq. in.}$$

The greatest possible error is $\frac{49}{64}$ sq. in. and the precision, which is the total range about $8\frac{15}{16}$ within which the true answer must occur, is $\frac{49}{32}$ sq. in. (i.e. twice the greatest possible error). The important thing to emphasize is the size of the possible error in the calculated answer. The average pupil who computes an area as above and gets $8\frac{15}{16}$ sq. in. is inclined to assume this is correct to the nearest sixteenth of a square inch, but the calculation shows that the possible error is much greater than this.

The intention in these problems is to promote a healthy skepticism about the accuracy of calculated answers, i.e. to have the students realize that working with approximate figures not only give approximate results but that the errors may be greatly expanded and that the results are generally not as good as they look. As a matter of fact they are practically never as good as the form of the answer suggests.

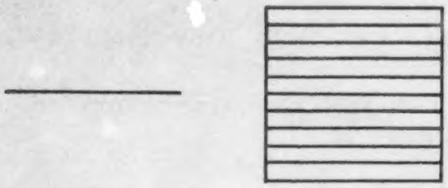
A question occasionally asked by alert students is along the following lines. We have emphasized that we are measuring the in-

*teriors* of rectangles. Does this mean that in laying on the unit squares to cover a rectangle we consider that we are using only the interior points of the squares, or do we think of the boundary line as belonging to the unit squares? In either case we seem to be in trouble, for consider the following rectangle broken up into unit squares:

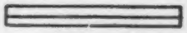


If we consider the boundaries of the squares as omitted, then all the points of the segments interior to the rectangle have not been covered, so we have not yet covered all the interior points of the rectangle. On the other hand, if we consider the squares to contain their edges then each of the interior edges is counted *twice* since it belongs to *two* unit squares and in addition we have covered not only the interior of the original rectangle, but the points of the rectangle itself.

The real question here comes down to "What is the area of a segment?", for to correct our apparently doubtful expressions for area we need to add or subtract a certain number of areas of segments. For definiteness, consider a segment one unit long, and consider also a unit square of area.



Let the unit of area be sliced into, say, ten similar strips as shown. Any one of the strips has an area of $\frac{1}{10}$ of a unit of area. Any such strip can be used to cover the line segment, and in fact it is too big since the segment has zero width, so the strip sticks out on both sides.



Thus whatever number is to be assigned to represent the area of the segment, it must certainly be less than $\frac{1}{10}$. But now suppose we sliced the unit of area into 1000 strips or 1,000,000 strips. The same argument shows that the number denoting the area of a segment must be less than $\frac{1}{1000}$ and must be less than $\frac{1}{1,000,000}$, and indeed must be less than any quotient of two counting numbers. But the only number which is less than all these fractions is the number zero. Thus we must conclude the area of a line segment is zero. Thus the answer to the question is that it does not matter how many segments are omitted or doubly counted since their areas are zero.

Since a rectangle is made up of four line segments, its area is the sum of four zeros, and hence is zero. In fact the area of any polygon is zero. For this reason it is not necessary to distinguish between the area of the interior of a rectangle and the area of a rectangle plus its interior. Notice the relation of this concept to our earlier decision to speak always of the area of the interior of a rectangle rather than to use the expression "area of a rectangle" as is commonly done. For us the area of a rectangle is actually zero, though there seems no occasion to take time for any extended discussion of this.

## Answers to Exercises 7-7a

3. Equal
4. The other two sides are then 6 inches and 4 inches long, and the length of the perimeter is 20 inches.
5. The number of units in the other two sides are $\ell$ and $w$. The number sentence could be any of the following: $p = \ell + w + \ell + w$ or $p = 2\ell + 2w$ or $p = 2(\ell + w)$
6. 1200 ft.; 400 yds.
7. $2000.
8. 600 in. or 50 ft. or $16\frac{2}{3}$ yds. (if inches, feet, yards are units chosen).
9. 456 in. or 38 ft. or $12\frac{2}{3}$ yds. Does not matter where doorways are.

10. 12 ft. by 12 ft. - no; not enough fence.

8 ft. by 3 ft. - no; doesn't use all the fence.

8 ft. by 4 ft. - yes.

Any five of the following: 11 ft. by 1 ft., 10 ft. by 2 ft., 9 ft. by 3 ft., 8 ft. by 4 ft., 7 ft. by 5 ft., 6 ft. by 6 ft.

11. $19\frac{4}{9}$ yd.

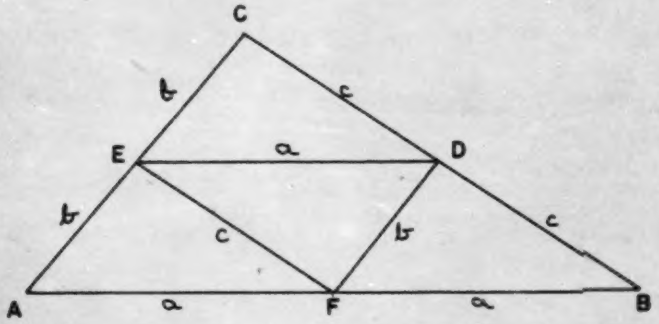
12. Any two of: $2\frac{3}{4}$ mi., 4840 yds., 14,520 ft., 174,240 in.

13. $687.50

14. He saved nothing.

*15. Side opposite given side is 40 ft. long, other two sides each 80 ft. long; either $2x + 2 \cdot 40 = 240$ or $2(x + 40) = 240$.

*16



| Simple closed curve | Number of units in perimeter |
|---|---|
| AFE | a + b + c |
| FBD | a + b + c |
| EDC | a + b + c |
| EFD | a + b + c |

| | |
|---|---|
| AFDE | 2a + 2b |
| EFBD | 2a + 2c |
| EFDC | 2b + 2c |
| ABDE | 3a + b + c |
| FBCE | a + b + 3c |
| DCAF | a + 3b + c |
| ABC | 2a + 2b + 2c |

## Answers to Exercises 7-7b

1. 11; 5; 55; $\frac{1}{16}$ sq. in.; $\frac{55}{16}$ sq. in. or $3\frac{7}{16}$ sq. in.

   The numbers 11 and 5 are the numbers of quarter inches in the length and width. Thus they are the numerators when the length of $2\frac{3}{4}$ is written as $\frac{11}{4}$ and when the width of $1\frac{1}{4}$ is written as $\frac{5}{4}$.

2. Small square $\frac{1}{2}$ inch by $\frac{1}{2}$ inch; 11; 9; $\frac{99}{4}$ or $24\frac{3}{4}$.

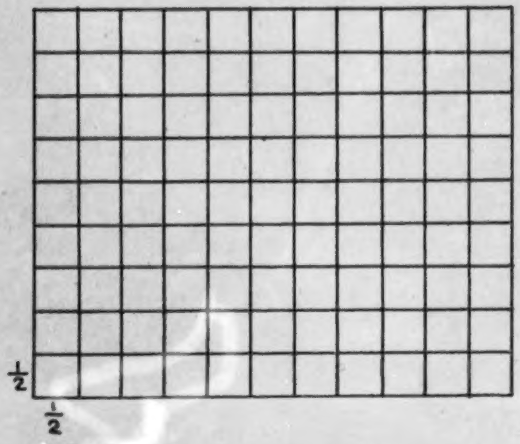
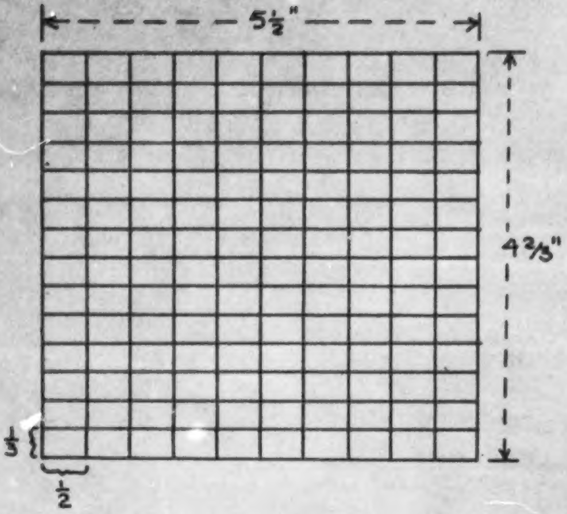


Figure not necessary to do problem. Use method of Problem 1 above.

*3.



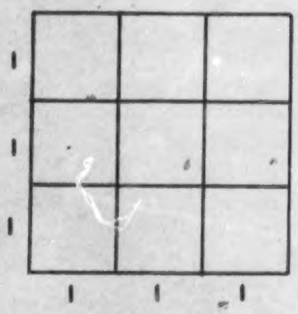
The division of the unit squares is suggested by the fractions in which measurements are given. Each rectangle is $\frac{1}{6}$ sq. in. There are $11 \times 14 = 154$ rectangles, so the area is $\frac{154}{6}$ sq. in. Note that $\frac{154}{6}$ is exactly $\frac{11}{2} \cdot \frac{14}{3}$.

4. The number of square units of area in the interior of a rectangle is the product of the number of units in the length by the number of units in the width.

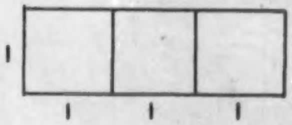
5. $A = \ell w$.

6. 1 square foot contains 144 square inches; 1 square yard contains 9 square feet.

7.



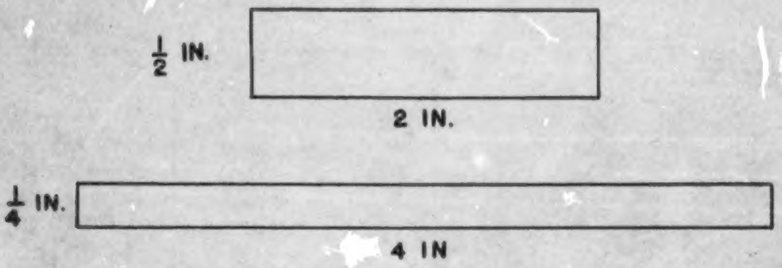
Three-inch square.



An area of three square inches.

The 3 inch square is larger. Its area is 9 square inches (These figures are not full size but to a scale with $\frac{1}{2}$ inch representing an inch).

8.



9. 108 sq. ft., 12 sq. yds., 15,552 sq. in.

10. 8100 sq. ft., 900 sq. yds.

11. Area is 3600 sq. ft. This is less than half the area of a baseball diamond. Actually it is $\frac{4}{9}$ of it.

12. 3,097,600 sq. yds. in 1 sq. mi.

13. Area of interior of smaller inside rectangle is $2\cdot 3 = 6$ square units.

Area of interior of larger inside rectangle is $6\cdot 3 = 18$ square units.

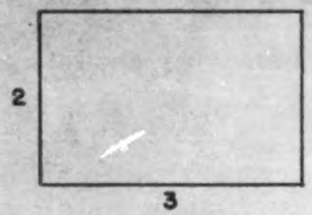
Area of interior of outside rectangle is $8\cdot 3 = 24$ square units.

$$24 = 6 + 18$$

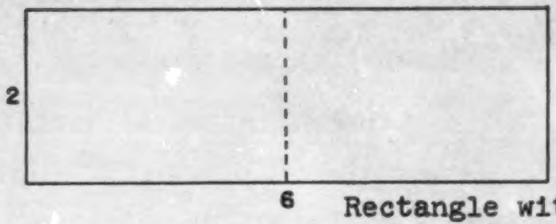
$$8\cdot 3 = 2\cdot 3 + 6\cdot 3$$

$$8\cdot 3 = (2 + 6)3 \qquad \text{Distributive property}$$

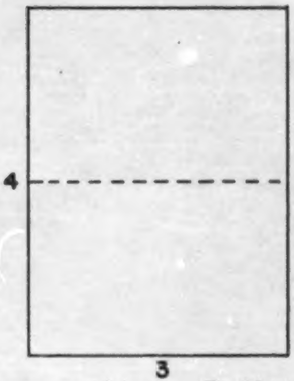
14.



Original rectangle.



Rectangle with doubled length.

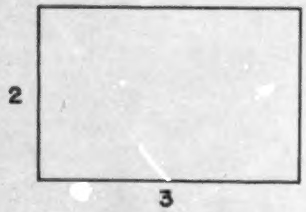


Rectangle with doubled height.

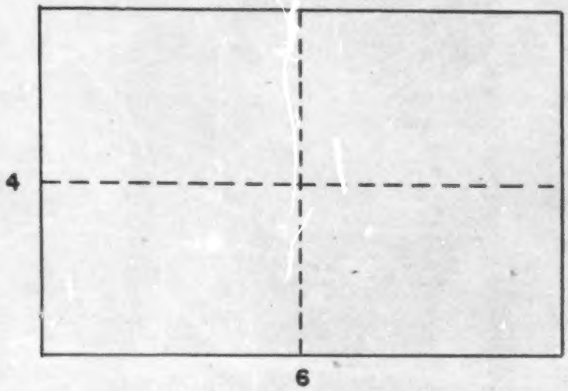
Original area is 6 square units. Both new areas are 12 square units. New areas are double the original one. The dotted lines are drawn to show geometrically why this is true. They are not part of the drawing asked for in the problem.

15. No. Doubling one measurement (length or width) of any rectangle doubles the area of its interior.

16.



Original rectangle.



New rectangle.

(Dotted lines not a part of required drawing, but are shown to make clear the geometric relationship).

Doubling both measurements (length and width) of any rectangle multiplies the old area by 4.

17. Perimeter of original rectangle is 10 units.

Perimeter of new rectangle is 20 units.

Doubling both measurements (length and width) of any rectangle doubles its perimeter. (The drawing of Problem 16 makes clear that this relation does not depend on the particular measurements.)

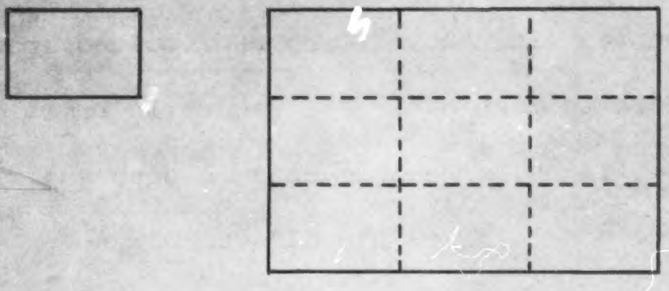
*18. $626 \cdot 422 = 2 \cdot 313 \cdot 2 \cdot 211 = 2 \cdot 2 \cdot 313 \cdot 211 = 2^2 \cdot 313 \cdot 211$

Commutative and associative properties are used.

The statement says the area of the interior of the new rectangle is 4 times that of the interior of the original rectangle ($4 = 2^2$).

Yes, it agrees with Problem 16.

*19. If length and width are tripled, the area of interior of the new rectangle is 9 times that of the interior of the original rectangle. ($9 = 3^2$)



If length is doubled and width tripled area of interior is multiplied by 6. $(6 = 2 \cdot 3)$

Answers to Exercises 7-7c

2. Area $\approx$ 2520 sq. in., $17\frac{1}{2}$ sq. ft., $1\frac{17}{18}$ sq. yd.
Perimeter $\approx$ 204 in., 17 ft., $5\frac{2}{3}$ yd.

3. Cost $\approx$ \$330.

4. Area $\approx$ $\frac{3}{25}$ sq. mi., 371, 712 sq. yd.

5. 14 lbs.

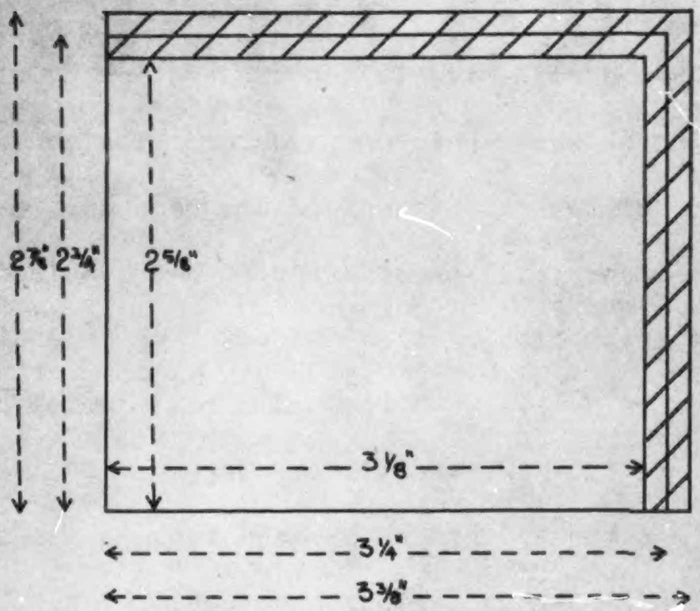
6. $2\frac{1}{2}$ sq. yds.

7. 99 sq. ft., (or 11 sq. yds.)

8. 570 sq. ft., $63\frac{1}{3}$ sq. yds. (or 82,080 sq. in.). It does not matter where the opening is.

9. $A = \ell w$, $A \approx \left(\frac{13}{4}\right)\left(\frac{11}{4}\right)$, $A \approx 8\frac{15}{16}$ so the area is approximately $8\frac{15}{16}$ sq. in. For figure see Problem 10.

10. $3\frac{1}{8}$ and $3\frac{3}{8}$; $2\frac{5}{8}$ and $2\frac{7}{8}$ ;



*11. Area of smallest rectangle is $8\frac{13}{64}$ sq. in.

Area of largest rectangle is $9\frac{45}{64}$ sq. in.

Difference between answer to Problem 9 and smallest possible true answer is $\frac{47}{64}$ sq. in.

Difference between answer to Problem 9 and largest possible true answer is $\frac{49}{64}$ sq. in.

Greatest possible error is therefore $\frac{49}{64}$ sq. in.

The area is $\left(8\frac{15}{16} \pm \frac{49}{64}\right)$sq. in.

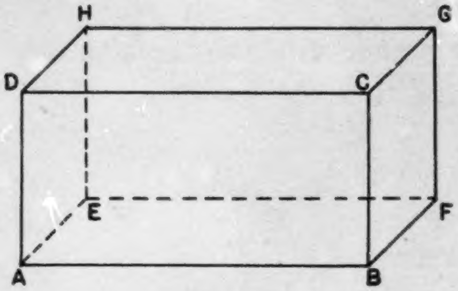
12. By Problem 17 of Exercises 7-7b the perimeter will be doubled by doubling length and width, so the order for the fence is correct. By Problem 16 of Exercises 7-7b the area of the garden will be multiplied by 4, so the fertilizer order is wrong. He should order 4 bags of fertilizer.

## 7-8. Rectangular Prism

The remarks under Section 7-7 about development of space perception should be repeated here only louder. The concept of volume and the sizes of different units of volume need much reinforcing. Models of cubic inches will be constructed but models of cubic feet and cubic yards are necessary also and practice in estimating volumes should be very helpful. A framework for a model of a cubic yard can be formed from 12 yardsticks. Most adults are astounded at the size of a cubic yard when they see one. One teacher had students bring in cardboard cartons that could be cut to the size of a cubic foot and assembled a model cubic yard from 27 of these boxes. Grocery cartons which are partitioned by cardboard dividers may be helpful in visualizing the subdivision of a volume into units.

The terms rectangular solid and rectangular prism have been used interchangeably with the thought that the first might be more meaningful at the beginning. The latter term is used more frequently, however. Just as a rectangle is composed of the points on its segments, so a rectangular prism is made up of the points on its faces, i.e. on its surface. Thus if a brick is suggested as an example of this figure, point out that the prism consists only of the surface of the brick. For this reason we do not speak of the volume of a rectangular prism but of the volume of the interior of a rectangular prism. As in the last section, the stating of a measurement requires both the number and the unit used, but any letter in a number sentence stands only for a number.

The rectangular prism with its 6 faces, 12 edges, and 8 vertices will need to be shown to the class with models and illustrations. The approach is frankly intuitive. However, once it has been agreed that the faces are rectangles, this can be used to deduce the fact that opposite faces have the same measurements. For example in the following figure



$\overline{EF}$ and $\overline{AB}$ have the same length because they are opposite sides of rectangle ABFE.

$\overline{AB}$ and $\overline{CD}$ have the same length because they are opposite sides of rectangle ABCD.

$\overline{CD}$ and $\overline{GH}$ have the same length because they are opposite sides of rectangle DCGH.

Thus the four segments $\overline{AB}$, $\overline{CD}$, $\overline{EF}$, $\overline{GH}$ all have the same length. Similarly $\overline{AE}$, $\overline{DH}$, $\overline{CG}$ and $\overline{BF}$ have the same length and $\overline{AD}$, $\overline{EH}$, $\overline{FG}$, and $\overline{BC}$ have the same length. This shows that any two opposite faces have the same measurements.

## Exercises 7-8a

These problems will probably be difficult, not because of the arithmetic, which is easy, but because of the spacial visualizing required. It is suggested that in doing these problems each pupil keep a box or other model in front of him to help visualize the problems that are presented. Except for the difficulty of visualizing, the problems are applications of work on rectangles in the last section. There is further drill on change of units. Problems 3 and 6 give further practice on finding areas by subtraction. Problem 9 introduces the term cube.

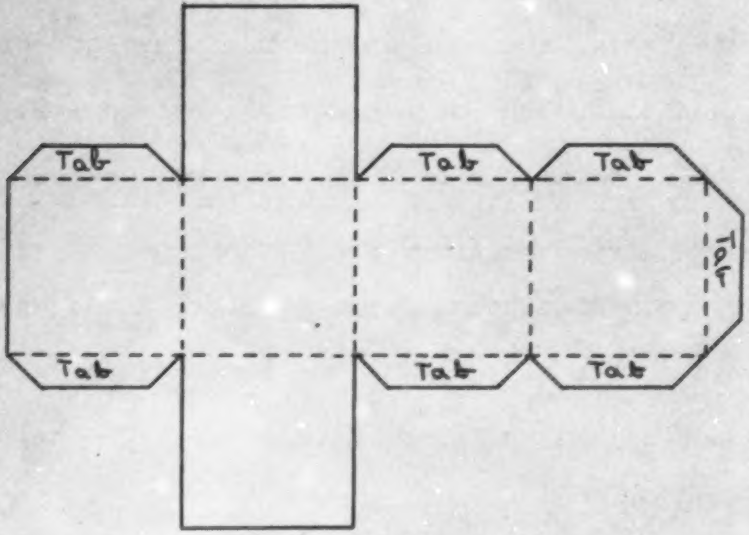
In discussing the choice of a unit of volume bring out that its edges are units of length and its faces units of area, while its interior is the unit of volume.

## Exercises 7-8b

In Problem 2 it would be desirable for the students to figure out their own patterns. If necessary, however, they may be given the pattern on the next page. Except for Problem 2, the problems of the set should be done largely or wholly in class. It is extremely important that the pupil see and be able to visualize that the effect of doubling the length of a rectangular prism amounts to laying two prisms end to end, both just like the original one. Similarly, that doubling two of the measurements amounts to putting four such prisms together and that doubling all three measurements

Pattern for 1 inch Cube

(Problem 2, Exercises 7-8b)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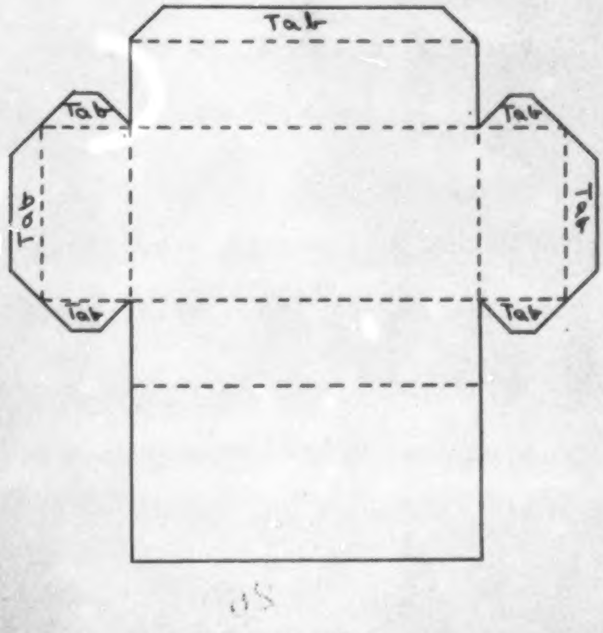
Pattern for Model of Problem 8, Exercises 7-8d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is equivalent to putting eight such prisms together. It is this visual perception, not a counting up of the cubes used, that is really significant. Also the models formed here show nicely the layers of cubes which are used in the next development.

In the question asked just before Exercises 7-8c, observe that if the base contains a half unit square, we need only slice a unit cube in two vertically and stand this half unit cube on the half unit square. Similarly, for a quarter unit square and so on. Thus the total number of cubes in the bottom layer of volume is always the same as the number of square units in the base.

## Exercises 7-8c

These problems have all been designed to emphasize the relation symbolized by the number sentence $V = Bh$. They have been deliberately chosen in such a way that the actual shape of the base is not known in any of them. This is to lay the foundation for the discussion later of volumes of interiors of prisms and cylinders, as well as the specific case of the rectangular prism which is discussed next.

In the question asked just before Exercises 7-8d the properties involved are the commutative and associative properties of multiplication.

## Exercises 7-8d

Problems 3 and 4 are an integral part of the development. Problem 5 lays the basis for changes of cubic units which are introduced frequently thereafter. Problem 7 is to clarify confusion of a 3 inch cube with a volume of 3 cubic inches. Problem 8 emphasizes two different possible shapes of a volume of 1 cubic inch. Patterns for the models, if this is necessary, are found on Page 253. Problem 9 is the algebraic approach to the geometric ideas developed in Exercises 7-8b. Problem 11 affords chance for more estimating and measuring. Problems 20-22 develop the ideas of greatest possible error and precision for computed volumes. They are comparable to Problems 9-11 of Exercises 7-7c and the same comments apply. With respect to error in computed volume, it is possible to give a graphic illustration of the substantial effect on volume of a fairly small error in measurements of length, width, and height as follows: Take a rectangular block of cheese (a two pound package possibly, though a smaller one would do) and imagine an error of $\frac{1}{8}$ inch. With a cheese cutter slice off $\frac{1}{8}$ inch thick slices from three faces which meet at a vertex. The total volume of these slices is the change in volume due to an error of only $\frac{1}{8}$ inch in each of the three measurements. This is quite an impressive amount. A rectangular block of clay could perhaps be used for the same purpose. This demonstration could be done without attempting to do Problems 20-22 above. As noted earlier, a skepticism with respect to the precision of such computed answers is a healthy thing.

In the discussion of dimension some of your better pupils may raise the question of describing the location of the sugar in other ways than by motions parallel to the edges of a room. For example in the second figure in the section under Dimension it may be suggested that the fly at A might simply point out the direction of S and tell his friend to crawl a certain distance in that direction. This is an excellent idea. However, note that these directions still call for two numbers, one describing the angle telling the direction in which the fly is to crawl, and the other giving the distance he must crawl. A precise definition of dimension involves very substantial difficulties beyond the scope of this course, but you will find that any "reasonable" way of describing location of points in the different sets will use the same number of numbers in the description, so that concept of dimension has meaning.

In the determination of volume for an irregular stone you may get various interesting suggestions to consider. One method is the immersion method, which would work as follows: Take a rectangular prism partially filled with water (or sand or salt or some other convenient substance). By measuring the length and width of the container and the depth of the water, the volume of water can be found. Now immerse the stone and determine the new volume. The difference of the volumes is the volume of the stone. In practice it is often a problem to find a suitable rectangular container. Sometimes toothbrush containers come in this shape and would do for small stones. Rectangular aquariums are good if you can find one and want to measure a good sized stone. If salt or sand is used as material plenty of cardboard boxes are available.

In the experiments to find the number of cubic inches in a liquid pint or in a bushel, the theoretical volume of the liquid pint is $28\frac{7}{8}$ cu. in. and a bushel is approximately 2150 cu. in. Possibly the bushel could be done by a few students as an outside project.

### Exercises 7-8a

1. 52 square units
2. 152 square inches
3. 230 sq. ft., $35\frac{5}{9}$ sq. yd.
4. Area of glass is 200 sq. in. = $1\frac{7}{18}$ sq. ft.

   Area of wood is 1100 sq. in. = $7\frac{23}{36}$ sq. ft.
5. (a) 180 squares of tile

   (b) 720 squares of tile
6. 396 sq. ft. No, it does not matter where windows are placed.
7. 3 quarts
8. 312 in., 26 ft., $8\frac{2}{3}$ yds.
9. 1944 sq. in. $13\frac{1}{2}$ sq. ft.

*10. $S = 2\ell w + 2wh + 2\ell h$ or $S = 2(\ell w + wh + \ell h)$

11. $e = 4\ell + 4w + 4h$ or $e = 4(\ell + w + h)$

12. There are 6 sq. ft. of surface to varnish on each box, so 300 sq. ft. to varnish in all. One pint of varnish will cover 72 sq. ft., so 2 qts. will cover 4 times as much or 288 sq. ft. This is not enough, so answer to problem is no.

## Exercises 7-8b

1. Cubic foot - interior of cube each edge of which is 1 ft.
Cubic yard - " " " " " " 1 yard.

3. 24

4. The number of cubes for each of the new solids is just double that in the original solid.

5. Doubling one measurement (length, width, or height) of any rectangular prism doubles the volume of its interior.

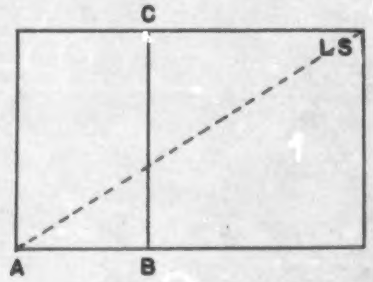
6. Ratio of number of cubes in new prism to number in original prism is $\frac{4}{1}$. That is, there are 4 times as many. The result would be the same if any two of the measurements are doubled.

7. Doubling any two measurements of any rectangular prism quadruples the volume of its interior (i.e. multiplies it by 4).

8. Ratio is $\frac{8}{1}$. That is, there are eight times as many.

9. Doubling all three measurements of any rectangular prism yields a prism whose interior has a volume 8 times that of the original prism.

10. BRAINBUSTER. The ant must crawl so that if the plates were flattened out into a single rectangle as shown his path would be a line segment.



## Exercises 7-8c

1. 85 cu. ft.

2. 20 cu. ft.

3. The number of cubic units of volume in the interior of a rectangular prism is the product of the number of square units of area in its base and the number of linear units in its height.

4. 2 ft.

5. 20 sq. ft., 2880 sq. in., $2\frac{2}{9}$ sq. yds.

If x stands for the number of square feet in the end (base) the number sentence is $13x = 260$.

6. Volume of room is 1600 cubic feet.

   30 children would require 1500 cubic feet, so 30 children is a legal number.

   Greatest legal number of children is 32.

7. $V = Bh$

## Exercises 7-8d

1. Volume is 748 cu. in.

   Area of wax paper is 343 sq. in.

2. *No*. Volume of 1st trunk is only 9 cu. ft.

   *Yes*. Volume of 2nd trunk is 10 cu. ft.

3. The number of cubic units of volume in the interior of a rectangular prism is the product of the numbers of linear units in the length, width, and height.

4. $V = \ell wh$

5. The number of inches in each edge of a cubic foot is 12, so the volume is $V = 12 \cdot 12 \cdot 12 = 1728$, or 1728 cu. in. There are 27 cubic feet in a cubic yard.

6. Volume is 27 cubic inches. No, if $\ell$ is less than 1 it is false.

7. $\frac{1}{2}$ inch

8. Yes, the volume *is* $23 \cdot 37 \cdot 59$ cubic inches.

   Volume of new prism is $46 \cdot 74 \cdot 118$ cubic inches.

$46 \cdot 74 \cdot 118 = 2 \cdot 23 \cdot 2 \cdot 37 \cdot 2 \cdot 59 =$

$2 \cdot 2 \cdot 2 \cdot 23 \cdot 37 \cdot 59 = 2^3 \cdot 23 \cdot 37 \cdot 59$

Second volume is $2^3$ or 8 times the first.

9. If all measurements are tripled the new volume will be 27 times the old volume ($27 = 3^3$).

If two dimensions are doubled and one tripled the new volume will be 12 times the old volume($12 = 2^2 \cdot 3$).

11. Weight $\approx$ 40,500 lbs. or $20\frac{1}{4}$ tons.

12. Volume $\approx$ 189,000 cu. ft., 7000 cu. yds.

13. 2 min.

14. $\frac{1}{16}$ mi. is the same as 110 yds. It is less than the distance between the end lines of a college football field (120 yds) but more than the distance between goal lines (100 yds). Number of people: 133,100.

15. Depth $\approx$ $1\frac{1}{2}$ ft., 18 in.

16. Weight of gold is 2250 lbs. Men can lift 2000 lbs., so they could not lift the chest.

17. 1728 in., 144 ft., 12 yds.

18. There would be space for 545,177,600 people. You could accommodate the population of the United States or of China, but not both at once. 1 cu. mi. = 5,451,776,000 cu. yds.

19. $183\frac{3}{4}$ cu. in.

20. $10\frac{1}{4}$ and $10\frac{3}{4}$; $4\frac{3}{4}$ and $5\frac{1}{4}$; $3\frac{1}{4}$ and $3\frac{3}{4}$.

21. $10\frac{1}{4}$ in. by $4\frac{3}{4}$ in. by $3\frac{1}{4}$ in. (smallest)

$10\frac{3}{4}$ in. by $5\frac{1}{4}$ in. by $3\frac{3}{4}$ in. (largest)

Volume of smallest prism is $158\frac{15}{64}$ cu. in.

Volume of largest prism is $211\frac{41}{64}$ cu. in.

Difference between smallest possible true volume, and answer to Problem 19 is $25\frac{33}{64}$ cu. in.

Difference between largest possible true volume and answer to Problem 19 is $27\frac{57}{64}$ cu. in.

Greatest possible error in answer to Problem 19 is, therefore, $27\frac{57}{64}$ cu. in. The volume is $(183\frac{3}{4} \pm 27\frac{57}{64})$ cu. in.

## Exercises 7-8e

2. 154 sq. ft. of carpet. Volume 1386 cu. ft.
3. Volume of pantry is 180 cu. ft., volume of deep freeze is 42 cu. ft. Remaining space is 138 cu. ft. or $5\frac{1}{9}$ cu. yd.
4. Doubling each dimension makes the new volume 8 times the original. The motor should have 8 times the power of the original motor.

## Exercises 7-8f

1. (a) 1 dim.
   (b) 2 dim.
   (c) 1 dim.
   (d) 2 dim.
   (e) 2 dim.
   (f) 3 dim.
   (g) 1 dim.
   (h) 2 dim.

(i) 2 dim.

(j) 2 dim.

(k) 3 dim.

(l) 3 dim.

(m) 2 dim.

(n) 3 dim.

## Exercises 7-8g

1. (a) $\frac{231}{4}$ cu. in. or $57\frac{3}{4}$ cu. in.

   (b) Yes

   (c) Often there is a roof-shaped top containing the pouring spout but not filled with milk.

2. Less than a quart by $2\frac{59}{64}$ cu. in. (Volume of container is $54\frac{53}{64}$ cu. in.)

3. The same volume of different materials have different weights. The old saying is roughly true for water.

4. (a) $64\frac{111}{128}$ cu. in. (Remember, however, this result is not as accurate as the form makes it look).

   (b) $67\frac{3}{8}$ cu. in.

   (c) No

5. (a) $35\frac{5}{32}$ cu. in.

   (b) $33\frac{11}{16}$ cu. in.

   (c) The box holds $1\frac{15}{32}$ cu. in. more than it should.

6. Presumably in measuring dry quantities such as berries and the like there are air spaces not filled with anything, so this is paid for by increasing the total volume which is to be called a quart.

7. Save \$ 1.07 by buying the bushel.

8. Save \$0.25 by buying the bushel.

9. Height should be $9\frac{3}{7}$ in. $\frac{7}{2} \cdot \frac{7}{2} \cdot h = \frac{231}{2}$

10. 11 in. by 7 in. by 3 in.

### 7-9. Computation with Measured Quantities

Students should not be left with the impression that length is the only item that is measured. This chapter has dealt largely with length, area and volume. Some time was spent on angle measure and students are familiar with time, temperature, and weight as measured quantities. The limitation of time forces a choice between a casual and inadequate treatment of many measures and a more thorough treatment of a few. A real understanding of measurement concepts should make any future work with other measures simpler.

Seventh grade students have had some experience adding and subtracting measured quantities. The explanation given here should be treated as review. Subtraction provides an excellent opportunity to strengthen the student's understanding of place value and the base of a number system. This can be brought out by showing that in subtracting measures, a procedure similar to subtracting numbers is used. In base ten, when a digit in the minuend is too small for the step in subtraction a one in the next larger place is exchanged for ten of the smaller. Similarly, when necessary in subtracting measures, one large unit is exchanged for its equivalent in smaller units. The complete lack of system in the relationships between various units of measure offers an opportunity to review number bases by showing what bases would be convenient to use for a specific set of two related units. This should appeal to the brighter student.

Multiplication and division of measured quantities will be new for most 7th grade students, but should cause little difficulty. Some of the better students will benefit from dividing composite quantities without changing to the smallest unit. An interpretation of the meaning of the remainder should be required. Practice in simple conversions is an essential part of this process, since remainders must be changed to the next smaller unit in order to complete the problem.

Some of these conversions provide considerable practice in fundamental arithmetic operations; units of area and volume require large numbers.

## Exercises 7-9

1. 10 hrs. 10 min.
2. 16 yds. 11 in.
3. 15 gal. 3 qts.
4. 28 hrs. 6 min. 8 sec.
5. 13 sq. yd. 3 sq. ft. 119 sq. in.

6. 11 cu. yd. 3 cu. ft. 807 cu. in.

7. 2 yd. 2 ft. 4 in.

8. 1 hr. 35 min.

9. 2 gal. 1 qt. 1 pt.

10. 2 yd. 2 ft. 7 in.

11. 2 sq. ft. 79 sq. in.

12. 6 cu. yd. 1717 cu. in.

13. (a) 22; 2 ft. 5 in.

(b) One ten was exchanged for 10 ones; one foot was exchanged for 12 inches. In base ten, "1" in any place can be exchanged for ten in the next smaller place. In measures, one large unit can be exchanged for its equal in smaller units.

14. 31 hr. 30 min.

15. 121 yd. 2 ft. 9 in.

16. 1146 gal. 3 qt.

17. 79 hr. 38 min. 45 sec.

18. 102 T. 804 lb.

19. 728 sq. ft. 46 sq. in. or 80 sq. yd. 8 sq. ft. 46 sq. in.

20. 47 min.

21. 2 yd. 1 ft.

22. 203 lb.

23. 2 qt. 1 pt.

24. 39 sq. in.

25. 58 cu. in.

Division without changing units. (No. 21)

$$\begin{array}{rlcl}
 & \text{2 yds.} & & \text{1 ft.} \\ \cline{2-4}
10\,) & \text{23 yds.} & & \text{1 ft.} \\
 & \underline{20} & & \\
 & \text{3 yds.} & = & \underline{9}\text{ ft.} \\
 & & & \text{10 ft.} \\
 & & & \underline{10}\text{ ft.}
\end{array}$$

$100_{\text{twelve}}$ sq. in. = 1 sq. ft.

$1000_{\text{twelve}}$ sq. in. = 1 cu. ft.

26. (a) Pints and quarts

(b) Feet and yards

(c) Ounces and pounds

(d) Hours and minutes

27. BRAINBUSTER. 3 oz. $\frac{3}{4}$ oz. is $\frac{1}{4}$ of the total weight; $4 \times \frac{3}{4} = 3$.

## Sample Test Questions

Select from this list. There are far too many questions for one test.

### I. True-False Questions

(F) 1. The counting numbers are all that are needed for both counting and measuring.

(F) 2. If two simple closed curves have the same length, then their interiors have the same size.

(T) 3. Perpendicular lines always meet so that rays with the endpoints at the intersection form right angles.

(T) 4. The interior of a rectangle can be used to measure the interior of a square.

(T) 5. All measuring units can be sub-divided.

(T) 6. A ruler is one form of a number line.

(F) 7. This number sentence would be used in changing 576 sq. in. to square feet.

$$576 \cdot \frac{1}{12} = n$$

(T) 8. Base 2 is related to the subdivisions of an inch on the ruler commonly used in school.

(F) 9. All rectangles are squares.

(F) 10. An obtuse angle is smaller than a right angle.

(T) 11. A measuring unit may have any size we choose.

(F) 12. If the interiors of two rectangular solids have the same size, then their lengths are the same and so are their widths and heights.

## II. Multiple Choice, Completion and Matching

(c) 1. To measure a line segment, you must use as a unit:

(a) An inch
(b) A foot
(c) A line segment
(d) A square inch
(e) None of these

(a) 2. Choose the best way to complete the statement, "Standard units of measure are used because:

(a) It is important for people to use the same unit in dealing with each other.
(b) Standard units give more accurate measurements than units which are not standard.
(c) People have always used them.
(d) They all fit base 10 numerals very well.
(e) None of these.

(b) 3. A hall is 6 feet long and $2\frac{1}{2}$ feet wide. How many square yards of carpet will cover it?

(a) 17
(b) $1\frac{2}{3}$
(c) 15
(d) $5\frac{2}{3}$
(e) None of these

(e) 4. If the measurement of an angle is $75^\circ$, the measurement of the exterior of the same angle is:

(a) $15^\circ$ (b) $95^\circ$ (c) $100^\circ$ (d) $105^\circ$
(e) None of these.

(e) 5. The interior of a 4-inch cube is:

(a) The same as 4 cu. in.

(b) Smaller than 4 cu. in.

(c) 16 times as large as 4 cu. in.

(d) 4 times as large as 4 cu. in.

(e) None of these

(e) 6. All rectangles with perimeters of 20 inches:

(a) Have the same interior measure.

(b) Have interior measures that increase as the base increases in length.

(c) Have interior measures that increase as the height increases in length.

(d) Have the same interior measure as a square with a 20 inch perimeter.

(e) None of these

(c) 7. The markings on a ruler divide each inch into 8 equal parts. The correct ways to report one measurement made with this ruler are:

(a) $3\frac{4}{16}$ and $3\frac{4}{16} \pm \frac{1}{32}$

(b) $3\frac{2}{8}$ in. and $3\frac{1}{4} \pm \frac{1}{8}$

(c) $3\frac{2}{8}$ in. and $3\frac{1}{4} \pm \frac{1}{16}$

(d) $3\frac{2}{8}$ in. and $3\frac{2}{8} \pm \frac{1}{8}$

(e) None of these

(b) 8. 2 cu. ft. and equal to:

(a) 24 cu. in. (d) 18 cu. yd.

(b) 3456 cu. in. (e) None of these

(c) 266 cu. in.

(d) 9. The rays on the complete scale for measuring angles are numbered:

(a) From 1° to 180° (d) From 0° to 360°

(b) From 0° to 180° (e) None of these

(c) From 1° to 360°

(b), (c) and (d) should be underlined. 10. Below is a list of things, some of which can be counted while others must be measured. Underline those which must be measured.

(a) Crowd at a meeting

(b) Time it takes to walk home.

(c) Weight of a brick

(d) Capacity of a gas tank

(e) Coats in your closet

11. Choose from the right-hand column, the term which best describes each term in the left-hand column and write its number on the line.

| | | |
|---|---|---|
| 7 | (a) Face of a cube ____ | 1. Ray |
| 2 | (b) Side of a rectangle ____ | 2. Line segment |
| 1 | (c) Side of an angle ____ | 3. Point |
| 3 | (d) Intersection of edges of a | 4. Line |

(Continued next page.)

rectangular prism ____ 5. Plane

6 (e) Face of a rectangular prism ____

6. Rectangle

7. Square

6, 12, 12. A rectangular prism has ____ faces, ____ edges, and
8 ______ vertices.

3 13. The interior of a rectangular prism is ____ dimen-
2 sional while any one of its faces is ____ dimension-
al.

$3\frac{3}{8}$ and 14. If a length is to be reported as $3\frac{2}{4}$ in., the true
$3\frac{5}{8}$ length must be between ______ in. and _____ in.

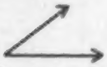
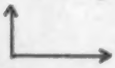
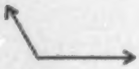
90 and 15. An obtuse angle is one whose measure is between ___
180 degrees and ______ degrees.

4 16. If the length and width of a rectangle are doubled,
the interior of the new rectangle is ____ times that
of the original one.

2 17. If the length and width of a rectangle are doubled,
the perimeter of the new rectangle is ____ times
that of the original one.

18. Which of the units in the right-hand column would
be the best to use to measure the thing listed in
the left-hand column? (A unit may be used more than
once.) Write the number of the unit in the right-
hand column on the line in the left-hand column.

| | | |
|---|---|---|
| 4 | (a) Air space in a room. ____ | 1. Degree |
| 3 | (b) Linoleum needed to cover a shelf. ____ | 2. Foot |
| | | 3. Square foot |
| 4 or 9 | (c) Amount of water in a small aquarium. ____ | 4. Cubic foot |
| | | 5. Mile |
| 2 or 7 | (d) A clothes line. ____ | 6. Square mile |
| 4 | (e) Space in a refrigerator. ____ | 7. Yard |
| 1 | (f) An angle ____ | 8. Square inch |
| | | 9. Cubic inch |

| | | |
|---|---|---|
| Protractor | 19. | The instrument used to measure an angle is called a ________________. |
| Nature | 20. | A unit of measure must have the same ________ as the object measured. |
| | 21. | The position of any point on a line can be described |
| 1 | | by, ____ number(s). The position of any point on a |
| 2 | | plane surface can be described by _____ number(s). |
| 3 | | The position of any point interior to a solid can be described by ____ number(s). |
| Acute | 22. |  represents a(n) ________ angle. |
| Right | |  represents a(n) ________ angle |
| Obtuse | |  represents a(n) ________ angle. |
| Any two lengths that add to 13'. Areas will vary. | 23. | Jim has 26 feet of left-over fencing to use around a small garden. Give two different sets of dimensions he could use. Find the area of the garden in |

each case.

24. A rectangular prism is 8 ft. long, 2 ft. wide and 3 ft. high. The area of the largest face is ______ sq. ft. The volume of the interior of the prism is ________ cu. ft.

24 sq. ft.

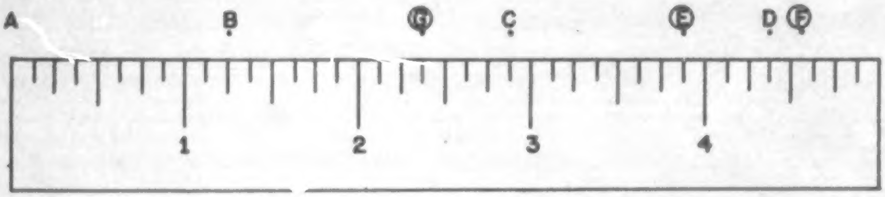
48 cu. ft.

25. An area is found to be 18 sq. ft. This is the same as ______ sq. in. or ______ sq. yd.

2592 sq. in.

2 sq. yd.

26\.



What number on the scale corresponds to:

(a) Point B — $1\frac{1}{4}$"

(b) Point C — $2\frac{7}{8}$"

(c) Point D — $4\frac{3}{8}$"

27. On the number scale for Problem 6, mark the point which corresponds to the numbers given below and label them with the letter indicated.

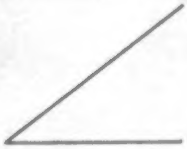
(a) $3\frac{7}{8}$ E

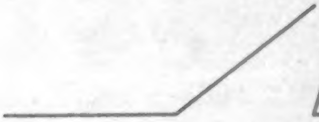
(b) $4\frac{9}{16}$ F

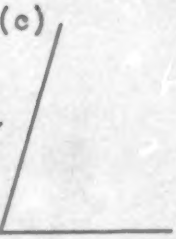
(c) $2\frac{3}{8}$ G

28. Measure these segments to the nearest 16th inch. Report the results so that the correct precision is indicated.

| | | |
|---|---|---|
| $3\frac{4}{16}$" or<br>$3\frac{1}{4}$" $\pm\frac{1}{32}$" | | (a) ________________________ |
| $4\frac{0}{16}$" or<br>4" $\pm\frac{1}{32}$" | | (b) ______________________________ |
| $1\frac{8}{16}$" or<br>$1\frac{1}{2}$" $\pm\frac{1}{32}$" | | (c) ___________ |
| | 29. | Measure each of these angles. |
| (a) 38° | | (a) |
| (b) 142° | | (b) |
| (c) 74° | | (c) |







| | | |
|---|---|---|
| | 30. | (a) Draw a rectangle $2\frac{3}{8}$ in. by 3 in. |
| $10\frac{3}{4}$" | | (b) Find the perimeter of this rectangle. |
| $7\frac{1}{8}$ sq. in. | | (c) Find its area. |
| 10 cu. ft. | 31. | A sand box is 3 feet wide and 4 feet long. How many cubic feet of sand are needed to fill the box to a depth of 10 in.? |
| 9 in. | 32. | An aquarium is 14 inches wide, 22 inches long and holds 12 gallons of water. How deep is the water? (1 gal. = 231 cu. in.) |
| | *33. | A rectangular playground is 180 ft. by 330 ft. |
| 59,400 sq. ft.<br>6,600 sq. yd. | | (a) What is its area in square feet? In square yds? |

1020 ft. (b) What is its perimeter?

$5,940 (c) What would it cost to put blacktop on the playground at 90¢ a square yard?

$1552.50 (d) A fence is to be put around the two short sides and one long side of the playground. What would this fence cost at $2.25 per foot?

34. A chest is 30 inches wide, 2 feet high, and 5 feet long.

55 sq. ft. (a) Find the area of the surface.

No (b) A small can of stain will cover 30 square feet. Is one small can enough to stain the top and sides?